# SHAPE CONTROL OF CHARGED SPACECRAFT CLUSTER WITH TWO OR THREE NODES 

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In 2002, King, Parker, Deshmukh and Chong presented a technical report introducing the idea of using electrostatic forces in spacecraft formation flying. This was the birth of the Coulomb formation flying concept. Since then, many areas related to Coulomb formation flying have been studied, such as the equilibrium solutions for a static multiple-craft Coulomb formation, the equilibrium solutions for a spinning twoand three-craft Coulomb formation, Coulomb virtual tether control, and hybrid formation flying control et. al. This dissertation investigates two aspects related to the shape control of a Coulomb cluster: two-craft collision avoidance using only Coulomb forces; two- and three-craft Coulomb virtual structure control.

A Lyapunov-based nonlinear feedback control and an open-loop patched-conicsection trajectory programming algorithm are developed to achieve the instant collision avoidance of two spacecraft. The Lyapunov-based control requires only separation distance and rate as feedback, the control achieves collision avoidance and retains the relative kinetic energy level. The trajectory programming algorithm searches a threephase patched-conic-section trajectory to avoid a potential collision. This approach achieves collision avoidance, retains the direction and magnitude of the relative velocity. There is an extra degree of freedom which can be utilized to find an optimal trajectory corresponding to a specific cost function.

On the side of Coulomb virtual structure control, at first a Lyapunov-based partial state feedback control is developed to control the separation distance of a spinning two-craft formation to the desired distance. The boundaries of the separation distance error due to the lack of the full position vector measurements are found analytically.

The study of the one-dimensional constraint three-craft Coulomb virtual structure control develops two approaches to solve the charge implementation issue. Then a switched Lyapunov-based control strategy is developed to stabilize the shape of a three-craft formation to the desired triangular configuration. The stability of the switched control is ensured using multiple Lyapunov function analysis tool. In the end a nonlinear control strategy is presented to stabilize the three-craft formation to a desired collinear configuration. The collinear configuration control does not require high-frequency switching of the control charges.

## DEDICATION

I dedicate this dissertation to my family, especially..
to my father for instilling me the scientific spirit;
to my mother for persistently concerning my life quality in the foreign country;
to my older brother for inspiring me to pursue the further studies abroad.

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## CHAPTER 1

## INTRODUCTION

The dissertation studies two aspects of research related to the shape control of a spacecraft formation using only Coulomb forces: Coulomb virtual structure control and two-spacecraft collision avoidance. This chapter introduces some important concepts related to the content of the dissertation, and also reviews related previous work.

### 1.1 Close-Proximity Flying

A close-proximity flying cluster of spacecraft is a collection of physically separated spacecraft (also called "distributed system of satellites") with their states coordinated to achieve certain objectives. Currently, close-proximity flying consists of three categories: spacecraft cluster, rendezvous and docking and spacecraft formation flying. The spacecraft formation flying concept has a tight tolerance on the relative configuration, while a spacecraft cluster has a relatively loose configuration. By altering the configuration of the formation, a wider range of missions can be accomplished as compared to an equivalent single, large spacecraft. Formation flying can even achieve some structural properties by using the concept of "virtual structure". Thus it performs as a large structure without actually building a large, flexible and usually vulnerable structure in space. Figure 1.1 illustrates the concept of a loosely controlled swarm of spacecraft.

In recent years, multiple-spacecraft distributed systems have aroused more and more interests in the Aerospace Engineering area. A cluster of spacecraft flying in


Figure 1.1: Illustration of clustered flying spacecraft. (Original image from website: http://gizmodo.com/363617/boeing-to-design-new-darpas-networked-swarmspacecrafts)
close-proximity can reduce mission cost and increase the redundancy, reliability, performance and survivability of space missions. The spacecraft cluster diversifies the launch risks and thus greatly improves the robustness to launch failures. When encountering a component failure of a spacecraft, the spacecraft cluster provides the capability to replace that component with another nearby spacecraft without sacrificing that portion of functionality. A growing number of space missions are being designed with the purpose of studying and/or exploring the advantages of the distributed systems of satellites. Princeton Satellite Systems describes some spacecraft clusters and formation flying missions on their website through: ${ }^{1}$

Examples include the TICS, F6 and Orbital Express programs at DARPA, the DART, MMS, SIRA, MAXIM and TPF missions at NASA, the

[^0]Proba-3, Darwin and Cluster missions at ESA and commercial missions like OLEV. The applications range from automated rendezvous for equipment and fuel delivery, to long-duration precise formation flying of distributed sensors, which could enable the detection of distant Earth-like planets. A common thread for all such missions is the need to autonomously perform coordinated operations among multiple freeflying spacecraft.

### 1.2 Coulomb Formation Flying

Coulomb Formation Flying (CFF) is a novel approach to control a spacecraft formation. The concept of CFF was originally proposed by King et. al. in 2002. [1] It actively creates an electric field about the spacecraft and utilizes the internal electrostatic forces within the formation to control the relative motion of the spacecraft. Because the Coulomb force is at least inversely proportional to the square of the separation distance, it is applicable in controlling a formation with the separation distances


Figure 1.2: Illustration of a Coulomb Formation Flying.
within 100 m . Outside of this region, the Coulomb force is too small to be utilized for kilo-Volt levels of the spacecraft potentials. Considering separation distances within 100 m , the magnitude of the Coulomb force generated is at the level of mili-Newtons to micro-Newtons, which is gentle and precise. Figure 1.2 shows a scenario of CFF.

On each spacecraft, the charge is generated by continuously emitting charged particles, ions or electrons, from the spacecraft by using the particle accelerators comparable to the hardware of an ion engine. The ion engine is the common actuator used in the electric propulsion (EP) approach, which generates large exit velocities of the ion particles such that the spacecraft gains a net external force from the momentum exchange. With the CFF concept, comparable charge emission devices need to expel the charge particles at a velocity large enough to escape the local electric field. The force due to the momentum exchange effect is negligible and the mass of the expelled particles is so trivial that the Coulomb thrusting is usually referred to as "essentially propellantless". Since the Coulomb thrusting approach does not need to generate mass flow (based on the purpose of the momentum exchange mechanism) as in EP, the power consumption is much less than that of EP.

### 1.3 Coulomb Virtual Structure

A virtual structure is a cluster of spacecraft with a nominally fixed relative configuration. In a virtual structure, all spacecraft should appear frozen as seen from the rotating local orbit frame. Generally, this fixed-configuration type of formation cannot happen naturally without an active control, except for the leader-following type formation in a circular orbit. Feedback control is required to stabilize the configuration to the reference configuration. The concept of virtual structure is one of several approaches to the formation control problem. Other approaches include the leader-following and behavioral approaches. The virtual structure approach is very convenient to prescribe a coordinated behavior of the formation. Reference [2] proposes a novel idea of intro-


Figure 1.3: Illustration of a Coulomb virtual structure.
ducing the formation feedback from the spacecraft to the virtual structure control. In that work, the authors at first assume a rigid structure in the orbit, then use the inverse dynamics method to determine a feedforward reference control using conventional thrusters that can hold the spacecraft to stay in the rigid structure configuration. In the last step, a feedback control loop is utilized to stabilize the formation to the desired rigid configuration.

Coulomb virtual structure control is a special case of virtual structure control. It is special in that it uses only the Coulomb forces as the control input. The classical virtual structure has the formation assuming a rigid shape regardless of the orbital motion. Inverse dynamics are used to create a feedforward control to continually compensate for the non-desired relative orbital motion. In contrast, the Coulomb virtual structure assumes an ideal shape which is an equilibrium of the charged relative orbital
motion. If the spacecraft potentials are held to specific values, then the differential gravitational forces are perfectly canceled. While the traditional virtual structure control uses conventional inertial thrusters that are capable of generating forces in any direction, Coulomb forces always lie in the line-of-sight direction. The non-affine nature of the system dynamics of the Coulomb virtual structure is another factor that makes the control problem much more complicated than the traditional virtual structure control.

A portion of the research on Coulomb virtual structures focuses on finding the equilibrium solutions of the charges and the positions for the spacecraft forming a certain shape naturally [3-6]. Another branch of research on Coulomb virtual structures investigates the spinning Coulomb formation. Most of the previous works on Coulomb virtual structures do not have any feedback control to stabilize the system, but rather provide the guidances of the reference feedforward charges. Reference [7] is the first work that studies the control of the Coulomb virtual structure. It develops a feedback control based on a linearized model to stabilize a three-craft Coulomb formation to a collinear configuration. Prior to this dissertation work, only linearized shape control of a three-craft Coulomb structure has been studied.

The Coulomb virtual tether is a concept that is very similar to the Coulomb virtual structure concept. A Coulomb virtual tether is a formation with spacecraft connected by Coulomb forces instead of physical instruments like the traditional tethers. The Coulomb virtual tether controls the orientation and the length of the virtual tether, focusing on the collinear configuration. Coulomb virtual structure considers only the shape control of the formation, with the desired shape being an equilibrium of the charged relative motion.

### 1.4 Spacecraft Collision Avoidance

In any close-proximity flying space mission designs, the possibility for spacecraft to collide must be treated carefully due to the huge cost of an unexpected collision. For

CFF with separation distances ranging within 100 meters, the chance for the spacecraft to collide is even higher. Collisions can occur when the spacecraft within the cluster have certain control or sensor failures, or the cluster is lacking the guidance strategy to guarantee the collision avoidance among a large number of the cluster members. For long-term Earth-orbit missions, a collision can also occur when the influences of the orbital disturbances accumulate.

Preventing collisions has many challenges. First, the collision onset must be sensed with sufficient accuracy to warrant a corrective maneuver. Second, a control strategy must be developed to provide the required small corrective forces without causing plume impingement issues on neighboring satellites. Currently, the most common approach in dealing with spacecraft collision avoidance problem is to examine the collision probability of a formation and perform some velocity corrections to reduce the probability to a negligible level. This approach usually uses thrusters to achieve the velocity corrections. This approach is intended to deal with the long-term disturbances and works for a formation or cluster with large separation distances at the km level.

For a very tight formation mission such as rendezvous/docking or CFF with the separation distances ranging within $10-100 \mathrm{~m}$, the above approach is no longer suitable to promptly handle such a potential collision. Another approach, called the instant collision avoidance maneuver, can be applied in these missions. In this approach, once a potential collision is detected the control starts to maneuver the two or multiple spacecraft to prevent the potential collision. Because this approach is designed to directly prevent collision instead of reducing the collision possibility, it is more prompt and thus suitable for tight formation missions. In the instantaneous collision avoidance problem, the main requirement is to avoid collision, which means the separation distance should always be greater than a certain constraint value.

### 1.5 Literature Review

King et al. [1] originally discussed the novel method of exploiting Coulomb forces for formation flying in 2002. Since then the CFF concept has been investigated for several different mission scenarios.

Berrymann and Vasavada et al. develop methods in References [3-6] to develop the equilibrium charges and positions for multiple charged spacecraft formation flying. Reference [3] presents a numerical algorithm to find the steady-state equilibriums which freeze the satellite formation with respect to the rotating Hill frame. Reference [4] investigates the analytical solutions to determine the equilibrium solutions for 2-craft and 3 -craft Coulomb formations. Reference [6] develops an analytical approach to find the feasible equilibrium for a square-shaped 4 -craft Coulomb formation. Reference [5] presents necessary conditions for static and circularly-restricted formations. Note that these references do not have any feedback control to stabilize the system, but rather only provide the reference feedforward charges.

Natarajan et al. investigate the 2-craft Coulomb virtual tether control problem in References [8-11]. Reference [8] introduces a charge feedback law to stabilize the distance in a two-craft Coulomb tether formation based on a linearized model. The gravity gradient torque is exploited to stabilize the Coulomb tether formation around the orbit nadir direction. Reference [9] designs a hybrid feedback control using traditional thrusters always in the normal-to-tether direction to stabilize the attitude of the formation without causing plume impingement issues. Reference [11] develops a pure Coulomb tether control based on the linearized out-of-plane decoupled model. In Reference [11], though the out-of-plane motion is not controlled, bounds on the initial out-of-plane oscillation are deduced using the linearized out-of-plane motion model based on Bessel functions.

Schaub and Hussein study the spinning Coulomb virtual structure problem in

References [7,12,13]. Reference [12] studies the invariant shape solutions for a spinning three-craft Coulomb formation. It shows that only the collinear configuration and expanding equilateral triangle configuration can be invariant. Reference [13] introduces the spinning two-craft Coulomb tether concept. It is the first work that analyses the open-loop stability of a Coulomb tether with constant spacecraft charges, based on a linearized model. Assuming the Coulomb tether is flying in deep space, it shows that the relative motion is locally stable if the spacecraft separation distance is less than the Debye length, and the out-of-plane motion is always stable. Reference [7] studies the three-craft Coulomb tether control problem. Based on a linearized model, a feedback control strategy is developed to stabilize the three-craft Coulomb tether to the collinear relative equilibria. The nonlinear system converges to the neighborhood of the desired equilibrium, but due to the approximation using the linearization technique, the size of the convergence neighborhood is limited.

Other than the above references, which are closely related to the Coulomb virtual structure problem, Joe et al. introduce a formation coordinate frame which tracks the principal axes of the formation in Reference [14]. Lappas et al. in Reference [15] develop a hybrid propulsion strategy by combining Coulomb forces and standard electric thrusters for formation flying on the orders of tens of meters in GEO. Simulation results show that incorporating Coulomb forces into the control strategy brings more than 80 percent power savings for propulsion. Reference [16] proposes a $N$-craft Coulomb structure control strategy by utilizing three drone spacecraft. The drones are used only to assist controlling the $N$ main spacecraft.

Other than CFF, other promising techniques for close-proximity flying include Electric Propulsion (EP) [1] and Electro-Magnetic Formation Flying (EMFF) [17]. EP systems generate forces by expelling ionic plumes. However, the ionic plumes can disturb the motions of nearby spacecraft. Furthermore, the intensive and caustic charge plumes can also damage sensitive instruments. The EMFF method controls relative separation
and attitude of the formation by creating electromagnetic dipoles on each spacecraft in concert with reaction wheels. In contrast to the EP method, the Coulomb formation flying technique has no plume contamination issues. The Coulomb force field in a vacuum is also simpler to model (point charge model) than the electromagnetic force field (dipole model), and the strength only drops off with the square of the separation distance and not the cube, as with the electromagnetic force field. The fuel-efficiency of Coulomb thrusting is at least 3-5 orders greater than that of Electric Propulsion (EP), and typically requires only a few Watts of electrical power to operate [1]. This is an essential advantage in long-term space missions.

### 1.6 Challenges And Prospects Of Coulomb Thrusting

The previous sections provide an overview of how the Coulomb forces can be utilized in formation control problems and collision avoidance maneuvers. This section concludes the advantages and limitations of the Coulomb force. There are three major advantages that make Coulomb thrusting approach very attractive to researchers in the field of spacecraft clusters:
(1) Low power consumption. Coulomb thrusting is said to be at least 3-5 orders of magnitude more power efficient than EP [1].
(2) Essentially propellentless. The Coulomb thrusting mechanism is not based on momentum exchange, as in EP, but rather on generating an electric field. Thus it does not require the high velocity plume flow that traditional or EP thrusters do. The mass variation of the spacecraft is negligible, thus Coulomb thrusting is usually deemed to be "essentially propellentless".
(3) Clean thrusting method with essentially no plume impingement issues. Due to the negligible low mass-flow rate, exhaust plume impingement issues do not need to be considered [1].

These advantages make the Coulomb thrusting approach very appealing in long-term space missions. However there are several challenges in utilizing the Coulomb thrusting concept:
(1) The Coulomb force lies on the line-of-sight direction, leading to limited control authority.
(2) Plasma shielding effects make the Coulomb force impractically small when the separation distance is larger than the local plasma Debye length.
(3) The magnitude of the Coulomb force decreases at least quadratically as the separation distance increases.
(4) For a Coulomb formation with three or more spacecraft, the individual charges appear in a non-affine form in the dynamics of the system.

The first disadvantage implies that using only Coulomb forces (cluster internal forces), it is not possible to directly control the inertial orientation of the formation. To control the complete inertial motions, CFF usually cooperates with other sources of force and torque such as differential gravity, EP and traditional thrusters. The second and third disadvantages indicate that the Coulomb force usage is applicable only in very close-proximity formation flying. The distances between the spacecraft should be less than the local Debye length $\lambda_{\mathrm{d}}$. Though the application of the Coulomb thrusting approach is strictly restricted to very tight formation flying, its advantages which would enable long-term space missions at very low fuel costs and with high reliabilities (no corrosive plume impingement issues) make it a very promising approach in close-clustered formation flying.

The fourth item in the list of the challenges is not an issue for two-craft CFF, because the required line-of-sight force can always be generated using an infinity of real charges. When the number of the spacecraft is three or more, the non-affine nature of the
dynamical system makes the complexity of the control problem increase dramatically as the number of spacecraft increases. This causes a physical implementability issue when there are three or more spacecraft. This issue makes the control strategy and the stability analysis very complicated. The specific examples of this problem, and the approaches that the dissertation develops to solve this problem are presented in Chapters 6, 7 and 8.

Considering the advantages and the disadvantages of the Coulomb thrusting concept, it is suitable for long-term, tight-cluster space missions where the separation distances remain within 100 meters. It is especially efficient in controlling the relative motions, and thus the overall shape of the cluster. In missions that require the control of the inertial orientation of the formation, Coulomb thrusting can cooperate with other sources of force and torque to achieve the mission while reducing the power or the propellent consumption.

### 1.7 Dissertation Contents And Outline

This dissertation studies two aspects of the research related to the shape control of a charged spacecraft cluster: 2-craft collision avoidance using Coulomb forces and 2and 3 -craft Coulomb virtual structure shape control.

Collision avoidance is a general concern in a tightly clustered flying spacecraft with separation distances ranging from dozens to hundreds of meters. With CFF, where the separation distances are within 100 meters, the possibility of a collision is even higher. This motivates us to investigate collision avoidance control that can be directly applied in CFF.

As introduced in Section 1.3, Coulomb virtual structure control has rarely been investigated. The dissertation studies the following aspects of the Coulomb virtual structure control problem:
(1) Nonlinear control of a spinning two-craft Coulomb virtual structure in GEO orbit.
(2) Nonlinear control of a one-dimensional constrained Coulomb virtual structure.
(3) Nonlinear control of a three-craft Coulomb virtual structure to an expected triangular shape (non-equilibrium configuration) in free space.
(4) Nonlinear control of a three-craft Coulomb virtual structure to an expected collinear shape (equilibrium configuration) in free space.

The dissertation follows eight chapters. Chapter 2 introduces the Coulomb thrusting concept in the plasma environment. Chapters 3 and 4 present a Lyapunov-based feedback control and a patched-conic-section trajectory programming algorithm, respectively, to achieve collision avoidance. Chapter 5 studies two-craft spinning Coulomb virtual structure control in the geosynchronous orbit (GEO). Chapter 6 investigates one-dimensional constrained three-craft Coulomb virtual structure control. Chapter 7 presents a stable switched controller for a three-craft triangular Coulomb virtual structure. Chapter 8 studies three-craft Coulomb virtual structure control problem with the desired configuration to be collinear. Chapter 9 concludes the dissertation.

## CHAPTER 2

## SPACECRAFT CHARGE CONTROL

### 2.1 Coulomb Force in a Plasma Environment



Figure 2.1: Illustration of Coulomb force generation.

The Coulomb force is also known as the electrostatic force. This dissertation focuses on utilizing Coulomb forces between spacecraft to control the relative motions of spacecraft within a cluster or formation. A spacecraft's charge level is actively generated and controlled by continuously emitting charged particles from the spacecraft. Devices comparable to the ion engine are used to expel charged particles such as ions or electrons. With the CFF concept, the velocity of the expelled mass flow does not have to be very large as with the EP approach. The velocity just needs to be large enough for the emitted charge particles to escape the local electric field. Note that an extra velocity of
the charge flow should be maintained to compensate for the inverse charge flow from the surrounding plasma environment to the spacecraft. For example, suppose a spacecraft is generating a positive charge. The electrons surrounding the spacecraft are attracted. This results in another charge flow that reduces the charge level of the spacecraft. To maintain the charge level of the spacecraft, the undesired charge flow due to the plasma environment should be compensated.

Because the Coulomb thrusting method requires only very low current levels, the power consumption in generating Coulomb thrusting is 3-5 orders magnitude less than the power required in EP [1]. The amount of emitted charge particle mass is trivial comparing to the mass of the spacecraft. Coulomb thrusting is said to be "essentially propellentless". Coulomb thrusting does not use propellent thus won't create plume impingement around the formation which might cause corrosion to nearby spacecraft using traditional thrusters.

In the space plasma environment, the electric field around a charged spacecraft can be derived from the electric potential of the spacecraft. The electric potential of a charge particle in plasma environment is given by: [18]

$$
\begin{equation*}
V=k_{\mathrm{c}} \frac{q}{r} \exp \left(-\frac{r}{\lambda_{\mathrm{d}}}\right) \tag{2.1}
\end{equation*}
$$

where $k_{\mathrm{c}}=8.99 \times 10^{9} \mathrm{C}^{-2} \cdot \mathrm{~N} \cdot \mathrm{~m}^{2}$ is Coulomb constant, $q$ is the charge of the particle, $r$ is the distance from the particle, $\lambda_{\mathrm{d}}$ is the Debye length. The Debye length controls how rapidly the space plasma will shield a charged object. Taking a gradient of $V$, the electric field is found:

$$
\begin{equation*}
\boldsymbol{E}=-\nabla_{\boldsymbol{r}} V=k_{\mathrm{c}} \frac{q}{r^{2}}\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right) \exp \left(-\frac{r}{\lambda_{\mathrm{d}}}\right) \hat{\boldsymbol{e}}_{\boldsymbol{r}} \tag{2.2}
\end{equation*}
$$

where $\hat{\boldsymbol{e}}_{\boldsymbol{r}}$ is the unit vector pointing from the particle to the position being considered.
This dissertation assumes that in a Coulomb cluster or formation the dimensions of the spacecraft are negligible compared to the size of the formation. This indicates that
in modeling the Coulomb forces the spacecraft in a CFF can be treated as particles, or homogeneously charged spheres. Based on the electric field expression in Eq. (2.2), the Coulomb forces within a cluster of charged spacecraft are found. Taking a $N$-spacecraft Coulomb cluster as an example, the total Coulomb force exerted onto the spacecraft- $i$ is given by

$$
\begin{equation*}
\boldsymbol{F}_{\mathrm{C}, i}=\sum_{j=1, j \neq i}^{N}-k_{\mathrm{c}} \frac{q_{i} q_{j}}{r_{i j}^{2}}\left(1+\frac{r_{i j}}{\lambda_{\mathrm{d}}}\right) e^{-\frac{r_{i j}}{\lambda_{\mathrm{d}}}} \hat{\boldsymbol{e}}_{i j} \tag{2.3}
\end{equation*}
$$

where $q_{i}$ is the charge of $i^{\text {th }}$ spacecraft, $r_{i j}$ is the distance between the $i^{\text {th }}$ and $j^{\text {th }}$ spacecraf, $\hat{\boldsymbol{e}}_{i j}$ is the unit vector pointing from $i^{\text {th }}$ spacecraft to $j^{\text {th }}$ spacecraft.

For an individual Coulomb force, which means only two spacecraft are considered, the magnitude of the Coulomb force is proportional to the charge product $Q_{i j}=q_{i} q_{j}$, and approximately inversely proportional to the square of the separation distance. Individual Coulomb forces alway lie on the line of sight direction and the direction is determined by the sign of the charge product $Q_{i j}$ due to the craft point mass or homogeneously charged sphere assumptions.

### 2.2 Space Plasma Environment

Though the Earth is an electrically neutral substance, more than $99 \%$ of matter in Universe exists in plasma state. Earth is also surrounded by plasma environment though its surface is neutral.

Plasma is the forth phase of matter that has enough energy for electrons to escape from the nucleus. It consists of randomly moving electrons and nuclei, in other words, charged particles. Plasma is greatly influenced by both magnetic and electric forces, and in turn, plasma particles affect the magnetic and electric fields. There are many interesting interactions between Earth's plasma and solar activities, however, these are not the focus of this dissertation. Because the plasma environment influences Coulomb forces, this dissertation briefly discusses the interaction of the plasma environment with
a charged spacecraft.
One important characteristic length of a plasma is the "Debye-Hückel shielding distance", $\lambda_{\mathrm{d}}$, which already appeared in Eqs. (2.1)-(2.3). Usually this length termed the Debye length for short. The of $\lambda_{\mathrm{d}}$ in terms of plasma properties is given by: $[19,20]$

$$
\begin{equation*}
\lambda_{\mathrm{d}}=\sqrt{\frac{\epsilon_{0} k T}{n_{0} e^{2}}} \tag{2.4}
\end{equation*}
$$

where $\epsilon_{0}=8.854 \times 10^{-12} \mathrm{farad} / \mathrm{m}$ is the permittivity of vaccuum, $k=1.381 \times 10^{-23} \mathrm{~J} / \mathrm{K}$ is the Boltzmann's constant, $T$ is the space plasma temperature in Kelvin, $n_{0}$ is the density of the undisturbed field particles, $e=\operatorname{sgn}(e) 1.602 \times 10^{-19} \mathrm{C}$ is the electron charge. The physical meaning of the Debye length is that only particles within $\lambda_{\mathrm{d}}$ are directly influenced by each others' electric fields. In order for a larger range of control using Coulomb forces, a sufficiently large value of $\lambda_{d}$ is required. From Eq. (2.4), it can be seen that $\lambda_{\mathrm{d}}$ increases as $T$ increases, and decreases as $n_{e}$ increases. The Debye length $\lambda_{\mathrm{d}}$ is determined by the environment, it is not a factor that can be actively controlled. But, based on knowledges and measurements from plasma physics field, the range of $\lambda_{\mathrm{d}}$ in specific situations can be obtained. Thus the guideline about dealing with the plasma's influence on CFF is achieved.

### 2.3 Plasma Shielding

Note that the expressions of the potential $V$, electric field $\boldsymbol{E}$ and Coulomb force $\boldsymbol{F}_{\mathrm{C}}$ in Eqs. (2.1)-(2.3) all contain the Debye length $\lambda_{\mathrm{d}}$. The Debye length $\lambda_{\mathrm{d}}$ represents the plasma shielding effect in space environment. Specifically, let's take the Coulomb force as an example:

$$
\begin{equation*}
\boldsymbol{F}_{\mathrm{C}}=-k_{\mathrm{c}} \frac{Q}{r^{2}}\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right) e^{-\frac{r}{\lambda_{\mathrm{d}}}} \hat{\boldsymbol{e}}_{r} \tag{2.5}
\end{equation*}
$$

This expression of the Coulomb force is valid only under the assumption that in the plasma environment the perturbing electrostatic potential is weak so that the electro-


Figure 2.2: Illustration of plasma shielding.
static potential energy is much less than the mean thermal energy, that is

$$
\begin{equation*}
e V(\boldsymbol{r}) \ll k_{\mathrm{B}} T \tag{2.6}
\end{equation*}
$$

Otherwise, a new expression has to be obtained and the derivations in this dissertation should be modified.

In this expression of the Coulomb force in Eq. (2.5), the factor $S\left(r / \lambda_{d}\right)=$ $\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right) e^{-\frac{r}{\lambda_{\mathrm{d}}}}$ quantifies the plasma shielding. To study the behavior of the plasma shielding, let us expand $S\left(r / \lambda_{\mathrm{d}}\right)$ into a series expression:

$$
\begin{align*}
S\left(r / \lambda_{\mathrm{d}}\right) & =\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right) e^{-\frac{r}{\lambda_{\mathrm{d}}}} \\
& =\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right)\left(1-\frac{r}{\lambda_{\mathrm{d}}}+\frac{1}{2!}\left(\frac{r}{\lambda_{\mathrm{d}}}\right)^{2}-\frac{1}{3!}\left(\frac{r}{\lambda_{\mathrm{d}}}\right)^{3}+\ldots\right) \\
& =1-\frac{1}{2}\left(\frac{r}{\lambda_{\mathrm{d}}}\right)^{2}+\frac{1}{6}\left(\frac{r}{\lambda_{\mathrm{d}}}\right)^{3}-\ldots \tag{2.7}
\end{align*}
$$

Figure 2.3 show the value of $S\left(r / \lambda_{\mathrm{d}}\right)$ under different values of $\frac{r}{\lambda_{\mathrm{d}}}$. It can be seen that $0<S\left(r / \lambda_{\mathrm{d}}\right) \leq 1 . ~ S\left(r / \lambda_{\mathrm{d}}\right)=1$ only when $r / \lambda_{\mathrm{d}}=0$ which implies $r=0$ or


Figure 2.3: Plasma shield curve, $S\left(r / \lambda_{\mathrm{d}}\right)$.
$\lambda_{\mathrm{d}} \rightarrow \infty . S\left(r / \lambda_{\mathrm{d}}\right)$ decreases to zero as $\frac{r}{\lambda_{\mathrm{d}}} \rightarrow \infty$. In CFF, we usually explore Coulomb forces within the range $0<\frac{r}{\lambda_{\mathrm{d}}}<1$. Note that $S(1) \approx 0.7358$. This indicates that in CFF the plasma shields at most $26.4 \%$ of the magnitude of a vacuum Coulomb force.

The typical values of the Debye length are listed in Table 2.1. [1] The Coulomb thrusting is applicable for a formation mission with separation distances between spacecraft within 100 m flying in GEO or High Earth Orbit (HEO), or within 40 m in deep space at 1 AU .

Table 2.1: Typical values of the Debye length.

| Location | Range of $\lambda_{\mathrm{d}}$ |
| :---: | :---: |
| LEO | $[0.02,0.4] \mathrm{m}$ |
| GEO | $[142,1496] \mathrm{m}$ |
| 1 AU in deep space | $[20,40] \mathrm{m}$ |

## CHAPTER 3

## TWO-SPACECRAFT COLLISION AVOIDANCE USING COULOMB FORCES WITH SEPARATION DISTANCE AND RATE FEEDBACK

This chapter considers the feedback control using only Coulomb forces to perform a collision avoidance maneuvers. A potential collision of the two spacecraft flying in deep space is considered where no external forces and torques are acting on the cluster. A charge feedback control strategy is investigated that maintains a desired minimum separation distance between two spacecraft. To minimize the sensor requirements, the control requires only the separation distances and the rates measurements between the craft during the collision avoidance phase. The separation distance is much simpler to measure than the full six Degree-Of-Freedom (DOF) relative state vector.

A very simple way to avoid a collision has both spacecraft charged up to large Coulomb values with equal sign. The resulting repulsive force drives the craft apart, thus avoiding the collision. But this strategy also results in the two spacecraft flying apart at a considerable velocity, thus noticeably changing their inertial motion. This can cause sensing issues for the spacecraft themselves, but is also of concern if the 2 craft are operating within a larger cluster of spacecraft. This additional velocity makes future collision avoidance maneuvers more challenging. Instead, the charge feedback control is developed with the additional goal to minimize changes to the relative kinetic energy level of the 2 spacecraft.

Finally, the chapter also considers the effect of charge saturation on the collision avoidance strategy. Even with sophisticated spacecraft designs, there will always be a physical limit to which a craft can safely be charged. Of interest is determining how much initial approach speed the craft can have and still avoid a collision if the charge levels are limited. Analytical conditions are investigated to guarantee that a collision can be avoided if a given charge limit is considered. The work in this chapter has been presented in Reference [21] and has been published and a journal article in Reference [22].


Figure 3.1: Illustration of the 2-spacecraft system.

### 3.1 Collision Avoidance Scenario

The spacecraft collision avoidance part in this dissertation focuses on a mission scenario where loosely clustered satellites are flying in deep space in a bounded configuration. The satellites are assumed to have a low approach speed with respect to each other, without external forces and torques acting on the cluster. The scenario being envisioned is that a potential collision of two spacecraft flying in deep space will happen in a short amount of time if no collision avoidance maneuver is applied. A control strategy that uses only Coulomb forces should be turned on and engaged to prevent this
potential collision. Figure 3.1 shows the scenario of the two-spacecraft system.
The primary requirements for collision avoidance control are to make the separation distance always greater than a certain restricted value and drive the distance outside of a certain potential collision region. To rigorously set up the problem, let's take a look at Figure 3.2. Spacecraft-1 (SC1) has a safety region

$$
\mathcal{B}_{s}=\left\{\boldsymbol{R} \mid\left\|\boldsymbol{R}-\boldsymbol{R}_{1}\right\| \leq r_{s}\right\}
$$

where $r_{s}$ is a constraint distance. $\mathcal{B}_{s}$ can never be penetrated. SC1 has another region called the activation region:

$$
\mathcal{B}_{a}=\left\{\boldsymbol{R} \mid\left\|\boldsymbol{R}-\boldsymbol{R}_{1}\right\| \leq r_{o}\right\}
$$

It is measured by the distance $r_{o}$. If Spacecraft-2 (SC2) enters the activation region $\mathcal{B}_{a}$ and is heading towards the safety region $\mathcal{B}_{s}$, then a collision avoidance maneuver is turned on to control the system to prevent the potential collision, and drive SC2 outside of the activation region $\mathcal{B}_{a}$.


Figure 3.2: Collision avoidance setup.

The primary collision avoidance goals are formulated as:

$$
\begin{align*}
& r(t) \geq r_{s}, \quad \text { for } t \geq 0,  \tag{3.1a}\\
& r\left(t_{f}\right)>r_{o}, \quad t_{f}<\infty \tag{3.1b}
\end{align*}
$$

Achievement of these goals results in a successful collision avoidance maneuver. If the inequalities in Eq. (3.1a) can not be satisfied by all means, then the potential collision is deemed as non-avoidable.

Because the relative motion is very important in CFF (the entire Coulomb structure control part in this dissertation studies the relative motion of CFF), we also expect that the relative kinetic energy level and keep the relative motion direction are retained. In summary, the collision avoidance maneuver has two primary requirements and two expected goals:

$$
\begin{align*}
r(t) & \geq r_{s}, \quad t \geq 0,  \tag{3.2a}\\
r\left(t_{f}\right) & >r_{o}, \quad t_{f}<\infty ;  \tag{3.2b}\\
\left\|\dot{\boldsymbol{r}}\left(t_{f}\right)\right\| & \approx\left\|\dot{\boldsymbol{r}}\left(t_{0}\right)\right\|,  \tag{3.2c}\\
\hat{\boldsymbol{e}}_{\dot{\boldsymbol{r}}}\left(t_{f}\right) & \approx \hat{\boldsymbol{e}}_{\dot{\boldsymbol{r}}}\left(t_{0}\right) \tag{3.2d}
\end{align*}
$$

Achievement of the expected goals in Eqs. (3.2c)-(3.2d) is a plus to the collision avoidance maneuver, but it's not required.

### 3.2 Charged Spacecraft Distance Equation Of Motion

Consider two spacecraft flying in the three-dimensional space where there are no external forces acting on the system as shown in Figure 3.1. In CFF concepts the electrostatic forces directly control separation distances $r_{i}$ but not the the inertial positions $\boldsymbol{R}_{i}$. This chapter intends to use the separation distance $r$ and the distance rate $\dot{r}$ as the control feedback, thus the separation distance equations of motion are required to develop the control strategy. The Coulomb force vector between the two
spacecraft, acting on $m_{1}$, is

$$
\begin{equation*}
\boldsymbol{F}=-k_{\mathrm{c}} \frac{q_{1} q_{2}}{r^{3}}\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right) e^{-\frac{r}{\lambda_{\mathrm{d}}} \boldsymbol{r}}=-k_{\mathrm{c}} \frac{q_{1} q_{2}}{r^{2}}\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right) e^{-\frac{r}{\lambda_{\mathrm{d}}}} \hat{\boldsymbol{e}}_{r} \tag{3.3}
\end{equation*}
$$

where $k_{\mathrm{c}}=8.99 \times 10^{9} \mathrm{C}^{-2} \cdot \mathrm{~N} \cdot \mathrm{~m}^{2}$ is the Coulomb constant, $r$ is the distance between the two spacecraft, $\boldsymbol{r}$ is the relative position vector pointing of spacecraft 1 (SC1) to spacecraft $2(\mathrm{SC} 2), \hat{\boldsymbol{e}}_{r}$ is the unit vector of $\boldsymbol{r}$, and $\lambda_{\mathrm{d}}$ is the Debye length. The effective range of a given electrical charge is smaller if the plasma Debye length is shorter. For high Earth orbits (HEO), the Debye length ranges between $100-1000$ meters $[1,23,24]$. CFF concepts typically have spacecraft separation distances ranging up to 100 meters.

The inertial equations of motion of the two spacecraft are

$$
\begin{align*}
& m_{1} \ddot{\boldsymbol{R}}_{1}=-k_{\mathrm{c}} \frac{q_{1} q_{2}}{r^{2}}\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right) e^{-\frac{r}{\lambda_{\mathrm{d}}}} \hat{\boldsymbol{e}}_{r}  \tag{3.4a}\\
& m_{2} \ddot{\boldsymbol{R}}_{2}=k_{\mathrm{c}} \frac{q_{1} q_{2}}{r^{2}}\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right) e^{-\frac{r}{\lambda_{\mathrm{d}}}} \hat{\boldsymbol{e}}_{r} \tag{3.4b}
\end{align*}
$$

where $\boldsymbol{R}_{i}$ is the inertial position vector of the $i^{\text {th }}$ spacecraft. The inertial relative acceleration vector $\ddot{\boldsymbol{r}}$ is

$$
\begin{equation*}
\ddot{\boldsymbol{r}}=\ddot{\boldsymbol{R}}_{2}-\ddot{\boldsymbol{R}}_{1}=\frac{k_{\mathrm{c}} q_{1} q_{2}}{m_{1} m_{2} r^{2}}\left(m_{1}+m_{2}\right)\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right) e^{-\frac{r}{\lambda_{\mathrm{d}}}} \hat{\boldsymbol{e}}_{r} \tag{3.5}
\end{equation*}
$$

In the kinematics of polar coordinates, the acceleration is given by

$$
\begin{equation*}
\ddot{\boldsymbol{r}}=\left(\ddot{r}-r \dot{\theta}^{2}\right) \hat{\boldsymbol{e}}_{r}+(2 \dot{r} \dot{\theta}+r \ddot{\theta}) \hat{\boldsymbol{e}}_{\theta} \tag{3.6}
\end{equation*}
$$

Substituting Eq. (3.6) into (3.5) yields the scalar separation distance equation of motion:

$$
\begin{equation*}
\ddot{r}=r \dot{\theta}^{2}+\frac{k_{\mathrm{c}} Q}{m_{1} m_{2} r^{2}}\left(m_{1}+m_{2}\right)\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right) e^{-\frac{r}{\lambda_{\mathrm{d}}}} \tag{3.7}
\end{equation*}
$$

Note that $2 \dot{r} \dot{\theta}+r \ddot{\theta}=0$ is a consequence of the inertial angular momentum being conserved with Coulomb forces. The term $Q=q_{1} q_{2}$ is the charge product between the two spacecraft charges $q_{i}$. Because only the separation distance and distance rate will be fed back to the controller, $\dot{\theta}$ should be expressed in terms of $r, \dot{r}$, and the
initial conditions. This is accomplished by considering the angular momentum about the cluster center of mass.

The position vectors $\boldsymbol{r}_{i}$ of the two spacecraft with respect to the center of mass are

$$
\begin{align*}
& \boldsymbol{r}_{1}=-\frac{m_{2} r}{m_{1}+m_{2}} \hat{\boldsymbol{e}}_{r}  \tag{3.8a}\\
& \boldsymbol{r}_{2}=\frac{m_{1} r}{m_{1}+m_{2}} \hat{\boldsymbol{e}}_{r} \tag{3.8b}
\end{align*}
$$

The angular momentum $\boldsymbol{H}_{c}$ of the system about the center of mass is

$$
\begin{equation*}
\boldsymbol{H}_{c}=\boldsymbol{r}_{1} \times \dot{\boldsymbol{r}}_{1} m_{1}+\boldsymbol{r}_{2} \times \dot{\boldsymbol{r}}_{2} m_{2}=\frac{m_{1} m_{2}}{m_{1}+m_{2}} r^{2} \dot{\theta} \hat{\boldsymbol{e}}_{3} \tag{3.9}
\end{equation*}
$$

Because there are no external torques acting on the system, the momentum vector $\boldsymbol{H}_{c}$ is conserved. From $\boldsymbol{H}_{c}=\boldsymbol{H}_{c}\left(t_{0}\right)$, the angular rate $\dot{\theta}$ is derived

$$
\begin{equation*}
\dot{\theta}=\frac{r_{o}^{2}}{r^{2}} \dot{\theta}\left(t_{0}\right)=\frac{r_{o}}{r^{2}}\left\|\dot{\boldsymbol{r}}\left(t_{0}\right)\right\| \sin \alpha_{0}=\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right) \frac{\left\|\boldsymbol{H}_{c}\left(t_{0}\right)\right\|}{r^{2}}, \tag{3.10}
\end{equation*}
$$

where $\alpha_{0}=\cos ^{-1}\left(\frac{\dot{\boldsymbol{r}}\left(t_{0}\right) \cdot \boldsymbol{r}_{o}}{\left\|\dot{\boldsymbol{r}}\left(t_{0}\right)\right\| r_{o}}\right)$ is the angle between $\dot{\boldsymbol{r}}\left(t_{0}\right)$ and $\boldsymbol{r}_{o}$. Thus the separation distance equations of motion in Eq. (3.7) is rewritten as

$$
\begin{equation*}
\ddot{r}=\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right)^{2} \frac{\left\|H_{c}\left(t_{0}\right)\right\|^{2}}{r^{3}}+\frac{\beta Q}{r^{2}}\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right) e^{-\frac{r}{\lambda_{\mathrm{d}}}}, \tag{3.11}
\end{equation*}
$$

where $\beta=\frac{k_{c}\left(m_{1}+m_{2}\right)}{m_{1} m_{2}}$. The collision avoidance control law challenge is to design the charge product $Q$ such that certain avoidance conditions are satisfied.

### 3.3 Unsaturated Control Law

Recalling the setup of the spacecraft collision avoidance problem, SC1 has a safe region $\mathcal{B}_{s}$ that can not be penetrated at any time. If another SC 2 enters the region $\mathcal{B}_{o}$ and is flying towards $\mathcal{B}_{s}$, this relative motion is deemed as a potential collision. A control law is then triggered to prevent the potential collision. Without loss of generality, it is assumed that the initial relative acceleration is zero. This assumption is reasonable
because upon detecting a potential collision, the spacecraft could equalize their charges in preparation for a collision avoidance maneuver.

The chief goals of the control are preventing the potential collision and driving SC 2 out of $\mathcal{B}_{r_{o}}$. They are formulated in Eqs. (3.2a)-(3.2b). Using only separation distance and distance rate as feedback, the Lyapunov control designed in this chapter is lacking information of the relative speed direction. This chapter only considers the first expected goal: maintain the kinetic energy level as formulated in Eq. (3.2c). Since only the separation distance is measured, not the full relative states, this condition will only achieve equal radial energy states.

If the trajectory of SC 2 does not touch the ball $\mathcal{B}_{r_{s}}$, no relative orbit correction is needed to avoid a collision. In this case the control strategy does not take effect. This situation is illustrated in Figure 3.3(a). Otherwise the electrostatic force fields are activated to repel the two spacecraft as shown in Figure 3.3(b).


Figure 3.3: Collision avoidance scenarios as seen by the first spacecraft.

Once SC 2 enters $\mathcal{B}_{o}$ and is moving towards $\mathcal{B}_{s}$, the collision avoidance control is triggered. The state $x_{1}=r(t)-r_{o}<0$ represents how far SC 2 has penetrated into the region $\mathcal{B}_{o}$, and the state $x_{2}=\dot{r}(t)+\dot{r}\left(t_{0}\right)$ represents the difference between the expected
radial departure rate and the actual distance rate (note that $\dot{r}\left(t_{0}\right)<0$ ). As stated above, the control law should reduce the absolute values of $x_{1}$ and $x_{2}$ when $r(t) \leq r_{o}$. When $r(t)>r_{o}$ a collision avoidance has been achieved. From here on the control is only trying to make $\dot{r}(t) \rightarrow-\dot{r}\left(t_{0}\right)$ to achieve the secondary goal, that is to maintain the radial relative kinetic energy level.

### 3.3.1 Lyapunov Based Control Design

Let us define the state vector $\boldsymbol{x}=\left(x_{1}, x_{2}\right)^{T}$ as

$$
\begin{align*}
& x_{1}=\left\{\begin{array}{ll}
r(t)-r_{o}, & r(t)<r_{o} \\
0, & r(t) \geq r_{o}
\end{array},\right.  \tag{3.12a}\\
& x_{2}=\dot{r}(t)+\dot{r}\left(t_{0}\right) . \tag{3.12b}
\end{align*}
$$

Any final radial separation distance $r_{\text {final }}>r_{o}$ is acceptable, and is reflected with a zero $x_{1}$ state. If the $2^{\text {nd }}$ spacecraft is outside of the region $\mathcal{B}_{r_{o}}$ and the radial departure rate is the opposite of the radial approach rate, then both collision avoidance states $x_{i}$ are zero. Thus the desired final states are $x_{1}\left(t_{f}\right)=0$ and $x_{2}\left(t_{f}\right)=0$. To avoid a collision, the safety region penetration variable $x_{1}(t)$ can never be less than $r_{s}-r_{o}$. To achieve this behavior the Lyapunov function penalizing $x_{1}$ is designed to go to infinity when $x_{1}(t)=r_{s}-r_{o}$. Let us define a Lyapunov candidate function as

$$
\begin{equation*}
V=\frac{1}{2} k_{1}\left(\frac{1}{x_{1}-r_{s}+r_{o}}-\frac{1}{r_{o}-r_{s}}\right)^{2}+\frac{1}{2} x_{2}^{2}, \tag{3.13}
\end{equation*}
$$

where $k_{1}$ is a constant positive coefficient. This function goes to infinity at the safety boundary $x_{1} \rightarrow r_{s}-r_{o}$ and if the radial separation rate grows unbounded. Note that even though $x_{1}$ is defined piecewise, it does not introduce a discontinuity in the Lyapunov function $V$ at $r(t)=r_{o}$. The first time derivative of the Lyapunov function is

$$
\begin{equation*}
\dot{V}=-k_{1}\left(\frac{1}{x_{1}-r_{s}+r_{o}}-\frac{1}{r_{o}-r_{s}}\right) \frac{\dot{x}_{1}}{\left(x_{1}-r_{s}+r_{o}\right)^{2}}+x_{2} \dot{x}_{2} . \tag{3.14}
\end{equation*}
$$

Note that here $\dot{x}_{2}=\ddot{r}$ as seen from Eq. (3.12b). The separation distance equation of motion in Eq. (3.11) relates the charge product $Q$ with $\ddot{r}$. To derive a control law from the Lyapunov function, $\dot{x}_{1}$ needs to be expressed in terms of the states, system constants and/or initial conditions. From the definition of $x_{1}$ in Eq. (3.12a), it is obvious that $\dot{x}_{1}=x_{2}-\dot{r}\left(t_{0}\right)$ when $r(t)<r_{o}$, but $\dot{x}_{1} \neq x_{2}-\dot{r}\left(t_{0}\right)$ when $r(t) \geq r_{o}$.

Note that the term $\left(\frac{1}{x_{1}-r_{s}+r_{o}}-\frac{1}{r_{o}-r_{s}}\right)$ is zero when $r(t) \geq r_{o}$, so the first term in Eq. (3.14) is zero when $r(t) \geq r_{o}$, no matter what $\dot{x}_{1}$ is. Thus $\dot{x}_{1}$ can be globally replaced with $x_{2}-\dot{r}\left(t_{0}\right)$ in the first term of Eq. (3.14) and simplify $\dot{V}$ to:

$$
\begin{equation*}
\dot{V}=-k_{1}\left(\frac{1}{x_{1}-r_{s}+r_{o}}-\frac{1}{r_{o}-r_{s}}\right) \frac{x_{2}-\dot{r}\left(t_{0}\right)}{\left(x_{1}-r_{s}+r_{o}\right)^{2}}+x_{2} \ddot{r}(t) . \tag{3.15}
\end{equation*}
$$

Note that $\dot{V}$ is continuous and well defined for all ranges of the separation distance $r$. Now the separation distance equation of motion in Eq. (3.11) can be directly substituted into $\dot{V}$ to design a charge feedback control law $Q$ using Lyapunov's direct method.

Assume a charge control law with the feedbacks of the separation distance and the separation distance rate as

$$
\begin{equation*}
Q=\left[\frac{k_{1}}{\beta}\left(\frac{1}{x_{1}-r_{s}+r_{o}}-\frac{1}{r_{o}-r_{s}}\right) \frac{r(t)^{2}}{\left(x_{1}-r_{s}+r_{o}\right)^{2}}-\frac{k_{2}}{\beta} r(t)^{2} x_{2}\right] \frac{1}{1+\frac{r}{\lambda_{\mathrm{d}}}} e^{\frac{r}{\lambda_{\mathrm{d}}}} . \tag{3.16}
\end{equation*}
$$

Using the Lyapunov function $V$ in Eq. (3.13), and substituting the charge control in Eq. (3.16) into the equations of motion in Eq. (3.11), differentiating $V$ yields the Lyapunov function rate expression:

$$
\begin{equation*}
\dot{V}=k_{1}\left(\frac{1}{x_{1}-r_{s}+r_{o}}-\frac{1}{r_{o}-r_{s}}\right) \frac{\dot{r}\left(t_{0}\right)}{\left(x_{1}-r_{s}+r_{o}\right)^{2}}-k_{2} x_{2}^{2}+x_{2}\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right)^{2} \frac{\left\|\boldsymbol{H}_{c}\right\|^{2}}{r^{3}} . \tag{3.17}
\end{equation*}
$$

Note that $\left(\frac{1}{x_{1}-r_{s}+r_{o}}-\frac{1}{r_{o}-r_{s}}\right) \geq 0$ and equals zero when $x_{1}=0$. Because $\dot{r}\left(t_{0}\right)<0$ the first term in Eq. (3.17) cannot be positive. Thus the Lyapunov function rate is bounded by

$$
\begin{equation*}
\dot{V} \leq-k_{2} x_{2}^{2}+x_{2}\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right)^{2} \frac{\left\|\boldsymbol{H}_{c}\right\|^{2}}{r^{3}} . \tag{3.18}
\end{equation*}
$$

Let the function $b(r)$ be defined as

$$
\begin{equation*}
b(r)=\frac{1}{k_{2}}\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right)^{2} \frac{\left\|\boldsymbol{H}_{c}\right\|^{2}}{r^{3}}>0 \tag{3.19}
\end{equation*}
$$

Note that $b(r) \rightarrow 0$ as $r \rightarrow \infty$. Because $b(r)>0$, Eq. (3.18) shows that $\dot{V}<0$ if

$$
\begin{equation*}
x_{2}>b(r) \quad \text { or } \quad x_{2}<0 . \tag{3.20}
\end{equation*}
$$

The $\dot{V}$ expression in Eq. (3.18) does not yet yield any stability guarantees. Initial conditions for $\dot{V}<0$ must be determined.

### 3.3.2 State Convergence And Collision Avoidance Achievement

Next the stability and convergence of the charge control law in Eq. (3.16) is discussed. A collision avoidance should result in $r(t)>r_{s}$, while a secondary goal attempts to drive $x_{2} \rightarrow 0$.

Theorem 1 For a two body system with equations of motion as shown in Eq. (3.4), the charge control law in Eq. (3.16) makes the state $x_{2}$ converge to the interval $[0, b(r)]$. Further, assuming $x_{1} \rightarrow 0$ in a finite time, $x_{2}$ converges either to 0 or to $b(r)$ as $t \rightarrow \infty$.

Proof Equation (3.20) shows that $\dot{V}<0$ if $x_{2}$ is outside of the interval $[0, b(r)]$. According to the Lyapunov stability theory, the charge control law in Eq (3.16) will drive $\dot{V}$ to zero. Thus $x_{2} \rightarrow[0, b(r)]$ asymptotically as $t \rightarrow \infty$.

By the asumption that $x_{1} \rightarrow 0$ in a finite time $t^{\dagger}$, the inequality in Eq. (3.18) becomes an equality for $t \geq t^{\dagger}$. Then $\dot{V}>0$ when $x_{2} \in(0, b(r))$, and $\dot{V}=0$ when $x_{2}=0$ or $x_{2}=b(r)$ for $t \geq t^{\dagger}$. According to the Lyapunov stability theory, $x_{2}$ will be driven to 0 or $b(r)$. So $x_{2}$ converges either to 0 or to $b(r)$ as $t \rightarrow \infty$.

Theorem 2 Assuming a two body system with the dynamics described by Eq. (3.4) is subjected to the charge control law in Eq. (3.16), then the states $\left(x_{1}, x_{2}\right) \rightarrow(0,0)$ as $t \rightarrow \infty$, where $x_{1}, x_{2}$ are defined by Eq. (3.12).

Proof Theorem 1 guarantees that $x_{2} \rightarrow[0, b(r)]$ as $t \rightarrow \infty$. The relationship between $x_{2}$ and $\dot{r}$ in Eq. (3.12b) indicates that $\dot{r} \rightarrow\left[-\dot{r}\left(t_{0}\right), b(r)-\dot{r}\left(t_{0}\right)\right]$. Because $\dot{r}\left(t_{0}\right)<0, \dot{r}$ will become a strictly positive value at a finite time $t^{+}$. As a result at time $t^{*}>t^{+}$ the separation distance reaches the outer collision avoidance distance $r_{o}$, and for $t>t^{*}$, $r(t)>r_{o}$. Refering to the definition of $x_{1}$ in Eq. (3.12), it can be concluded that $x_{1} \rightarrow 0$ in a finite time. Due to $\dot{r}$ being strictly positive, the separation distance $r \rightarrow \infty$ as $t \rightarrow \infty$.

Having shown that the 2 spacecraft will depart the collision avoidance region, next the convergence of $x_{2}$ is investigated as $t \rightarrow \infty$. If $\boldsymbol{H}_{c}=\mathbf{0}$, the interval $[0, b(r)]$ becomes the zero point. Thus $x_{2} \rightarrow 0$ due to the theorem 1 property $x_{2} \rightarrow[0, b(r)]$. For the case where $\boldsymbol{H}_{c} \neq \mathbf{0}$ the properties of $x_{2}$ need to be further investigated. The definition in Eq. (3.12b) yields $x_{2}\left(t_{0}\right)=2 \dot{r}\left(t_{0}\right)<0$. Here $x_{2}$ will either converge to 0 or to $b(r)$ because $x_{1} \rightarrow 0$ in a finite time has been proven. If $x_{2}$ never reaches zero, then $x_{2} \rightarrow 0$. If $x_{2}$ crosses zero and converges to $b(r)$, then $x_{2} \rightarrow 0$ due to $b(r) \rightarrow 0$.

Theorem 3 For a two body system with dynamics described by Eq. (3.4), the charge product control law in Eq. (3.16) prevents any potential collision by keeping $r(t)>r_{s}$ for all time, and making $r(t)>r_{o}$ in a finite time.

Proof While proving $x_{1} \rightarrow 0$ in theorem 2, it has been shown that $r(t)>r_{o}$ is true for $t>t^{*}$. Thus the condition $r(t) \geq r_{s}$ for all time is left to be proven. Note that $r(t)$ starts with $r_{o}>r_{s}$. The definitions of $V$ in Eq. (3.15) and $x_{1}$ in Eq. (3.12a) show that $V \rightarrow \infty$ if and only if $r(t)$ decreases to be $r_{s}$ or $x_{2} \rightarrow \infty$. Theorem 1 shows that $x_{2} \nrightarrow \infty$. Thus to prove $r(t)>r_{s}$ for all time, it's equivalent to prove that $V \nrightarrow \infty$ for all time.

The inequality in Eq. (3.18) shows that the only chance for $\dot{V}$ to be positive is $x_{2} \in(0, b(r))$. Thus a necessary but not sufficient condition for $V \rightarrow \infty$ is that $x_{2}$ stays in $(0, b(r))$ for an infinite time. But as mentioned while proving theorem $2, r(t)$ is
increasing as $x_{2} \in(0, b(r))$. Since $x_{1}$ increases as $r(t)$ increases, $x_{1}$ does not decrease to $r_{o}-r_{s}$ when $x_{2} \in(0, b(r))$. The definition of $V$ in Eq. (3.13) shows that $V$ is bounded when $x_{2} \in(0, b(r))$ and $x_{1}>r_{o}-r_{s}$. Thus even when $x_{2} \in(0, b(r))$ for an infinite time, $V$ is still bounded. So $V \nrightarrow \infty$ is guaranteed for all time, hence $r(t)>r_{s}$ is true for all time.

Practically speaking the range of the electrostatic control is limited due to the drop off of the Coulomb field strength with increasing separation distances. As a result, the controller will be turned off after the state goes inside a certain deadzone region. Let us define a radius $r_{c}>r_{o}$ where the collision avoidance charge control is turned off. The effect of this limitation is a termination of the control when $x_{1}=0$ and $r(t)>r_{c}$. Note that when the truncation happens, the potential collision has been avoided. After the control charges are turned off, there are no forces acting on the spacecraft. The two spacecraft are now flying freely in space (with the assumption that the spacecraft are flying in free space) with constant velocities. The separation distance rate is still bounded, even though it's not converging to the magnitude of the approach rate.

Given the charge product in Eq. (3.16) to produce the required electrostatic force field, the individual spacecraft charges $q_{i}$ are evaluated through

$$
\begin{align*}
& q_{1}=\sqrt{|Q|}  \tag{3.21}\\
& q_{2}=\operatorname{sign}(Q) q_{1} . \tag{3.22}
\end{align*}
$$

There is an infinity number of choices for how $Q$ can be mapped into $q_{1}$ and $q_{2}$. This strategy evenly distributes the charge amount across both craft. If one spacecraft can handle a higher charge level than the other spacecraft, adding a coefficient can adjust the charge distribution.

### 3.4 Saturated Collision Avoidance Analysis

Without the saturations of the spacecraft charges, the controller presented in the previous section can always prevent a potential collision. But in reality the spacecraft charge magnitudes are always limited. The ability of the two-body system to prevent a potential collision is reduced compared with the non-saturated control law. If the spacecraft are moving fast enough, then a collision cannot be avoided with a limited force. Hence for a given pair of limited charges, collision avoidance cannot be guaranteed for all initial conditions.

This section discusses limited charge control requirements for a collision to be preventable. Assume the two spacecraft are fully charged such that the charge product reaches its maximum positive value. If the separation distance $r$ still decreases to be less than the safety restraint distance $r_{s}$, the potential collision is deemed as not avoidable. Otherwise, the potential collision is avoidable.

### 3.4.1 Constant Charge Spacecraft Equations Of Motion

Our discussion of the conditions for a potential collision to be avoidable is based on the assumption that the charge product remains at its maximum value $Q=Q_{\text {max }}>0$ to generate the largest repulsive force. The Coulomb force expression in Eq. (3.3) simplifies to

$$
\begin{equation*}
\boldsymbol{F}=-k_{\mathrm{c}} \frac{Q_{\max }}{r^{3}}\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right) e^{-\frac{r}{\lambda_{\mathrm{d}}}} \boldsymbol{r} \tag{3.23}
\end{equation*}
$$

and the differential relative equation of motion is

$$
\begin{equation*}
\ddot{\boldsymbol{r}}=\beta \frac{Q_{\max }}{r^{3}}\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right) e^{-\frac{r}{\lambda_{\mathrm{d}}} \boldsymbol{r}} . \tag{3.24}
\end{equation*}
$$

Note that the form of the Coulomb force is very similar to the gravity force; this makes it possible to describe the motion using the formulas of the gravitational 2-body problem (2BP). Reference 12 provides an approach to analyze this Coulombforced spacecraft motion using 2BP method. To apply a 2 BP method in analyzing the

Coulomb-forced motion, it is necessary to find the radius and the energy equation in a similar form as in the 2BP. Let us introduce the effective gravitational parameter

$$
\begin{equation*}
\mu(r)=-k_{\mathrm{c}} \frac{Q_{\max }\left(m_{1}+m_{2}\right)}{m_{1} m_{2}}\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right) e^{-\frac{r}{\lambda_{\mathrm{d}}}} . \tag{3.25}
\end{equation*}
$$

Next, assume that $r \ll \lambda_{\mathrm{d}}$, which means the plasma shielding effect is negligible. Then $\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right) e^{-\frac{r}{\lambda_{\mathrm{d}}}}=1$ and the parameter $\mu(r)$ becomes a constant

$$
\begin{equation*}
\mu=-k_{\mathrm{c}} \frac{Q_{\max }\left(m_{1}+m_{2}\right)}{m_{1} m_{2}} . \tag{3.26}
\end{equation*}
$$

The relative equations of motion reduce to the familiar 2 BP form

$$
\begin{equation*}
\ddot{r}=-\frac{\mu}{r^{3}} r . \tag{3.27}
\end{equation*}
$$

Eq. (3.27) has the same form as the equation of motion of the gravitational 2BP, except that here $\mu$ is a negative number because $Q_{\max }>0$. By assuming $r \ll \lambda_{\mathrm{d}}, \mu$ becomes a constant, so the orbit radial trajectory is a conic section curve. Because $\mu<0$ for the repulsive force case, all relative trajectories are hyperbolas where craft 2 orbits the farther focus. [12] The signs of some parameters of the conic section are different from that of the gravitational 2BP. In our case $\mu<0$, the semi-latus radium $p<0$ and the semi-major axis $a>0$.

Because the repulsive hyperbolic motion has the craft orbit about an un-occupied focal point, the radial equation is different from that of the gravitational 2BP [12]:

$$
\begin{equation*}
r=\frac{p}{1-e \cos f} . \tag{3.28}
\end{equation*}
$$

Here the semi-latus rectum $p=h^{2} \mu<0$, and $h$ is the magnitude of the specific angular momentum $\boldsymbol{h}=\boldsymbol{r} \times \dot{\boldsymbol{r}}$. The energy equation is derived in the same procedure as the 2 BP , and yields an identical equation:

$$
\begin{equation*}
\frac{v^{2}}{2}-\frac{\mu}{r}=-\frac{\mu}{2 a} \tag{3.29}
\end{equation*}
$$

where $v$ is the magnitude of velocity vector $\dot{\boldsymbol{r}}$

$$
\begin{equation*}
v^{2}=\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}=\dot{r}^{2}+(r \dot{f})^{2}=\dot{r}^{2}+\frac{h^{2}}{r^{2}} \tag{3.30}
\end{equation*}
$$

and $\dot{f}$ is the in-plane rotation rate.
Because the total energy is positive, the relative trajectory of the two spacecraft is a hyperbola. As seen by $\mathrm{SC} 1, \mathrm{SC} 2$ is traveling along the hyperbola, and SC 1 is standing at the farther focus point [12] as illustrated in Figure 3.4.


Figure 3.4: Illustration of the 2-Body hyperbolic trajectory.

From Eq. (3.28), the closest separation distance corresponds to $r(f=0)$ that is the radius of periapsis

$$
\begin{equation*}
r_{p}=\frac{p}{1-e}=a(1+e) \tag{3.31}
\end{equation*}
$$

Thus, given an initial spacecraft approaching speed, finding the criterion for a collision avoidance is to determine a required saturated charge level that guarantees

$$
\begin{equation*}
r_{p} \geq r_{s} \tag{3.32}
\end{equation*}
$$

### 3.4.2 Avoidance Analysis

When the specific angular momentum satisfies $\boldsymbol{h} \neq \mathbf{0}$, there exists an offset distance $d$ between the position of SC 1 and the direction of the relative velocity of SC 2 ,
as shown in Figure 3.4. Note that here it is assumed that the current flight path will result in a potential collision where $r$ will become less than $r_{s}$. The specific angular momentum is represented in terms of $d$ and $v_{0}$ :

$$
\begin{equation*}
h=\left\|\boldsymbol{r}_{o} \times \dot{\boldsymbol{r}}\left(t_{0}\right)\right\|=d v_{0} . \tag{3.33}
\end{equation*}
$$

The parameter $h$ is expressed in terms of $v_{0}$ and $d$ instead of the $\dot{r}\left(t_{0}\right)$ and $\dot{f}_{0}$ set, because it will be easier to find out the criterion of $v_{0}$. Assume that $r(t), \dot{r}(t)$ and $f$ can be measured, $v_{0}$ and $d$ are calculated through:

$$
\begin{align*}
v_{0} & =\sqrt{\dot{r}\left(t_{0}\right)^{2}+\left(r_{o} \dot{f}_{0}\right)^{2}},  \tag{3.34}\\
d & =\frac{r_{o}^{2} \dot{f}_{0}}{v_{0}} \tag{3.35}
\end{align*}
$$

Because the angular momentum is conserved during the electrostatic collision avoidance maneuver as with the 2 BP , the relationship between the angular momentum and the orbit elements is:

$$
\begin{equation*}
h^{2}=\mu a\left(1-e^{2}\right) . \tag{3.36}
\end{equation*}
$$

Solving for the eccentricity $e$ yields

$$
\begin{equation*}
e=\sqrt{1-\frac{h^{2}}{\mu a}} \tag{3.37}
\end{equation*}
$$

This $e$ formulation can be used to calculate the periapses radius $r_{p}$ :

$$
\begin{equation*}
r_{p}=a(1+e)=a+\sqrt{a^{2}-\frac{a h^{2}}{\mu}} . \tag{3.38}
\end{equation*}
$$

The collision avoidance criterion $r_{p} \geq r_{s}$ yields the condition

$$
\begin{equation*}
a+\sqrt{a^{2}-\frac{a h^{2}}{\mu}} \geq r_{s} \tag{3.39}
\end{equation*}
$$

Subtracting $a$ from both sides and squaring the result yields

$$
\begin{equation*}
-\frac{a h^{2}}{\mu} \geq r_{s}^{2}-2 a r_{s} \tag{3.40}
\end{equation*}
$$

Now the semi-major axis $a$ is needed to obtain the relationship between $\mu$ and the initial states of the system. From the energy equation in Eq. (3.29), $a$ is solved as:

$$
\begin{equation*}
a=\frac{r_{o} \mu}{2 \mu-r_{o} v_{0}^{2}} . \tag{3.41}
\end{equation*}
$$

Substituting Eq. (3.41) into Eq. (3.40), and using $h=v_{0} d$, yields

$$
\begin{equation*}
-\frac{r_{o} v_{0}^{2} d^{2}}{2 \mu-r_{o} v_{0}^{2}} \geq r_{s}^{2}-\frac{2 r_{o} r_{s} \mu}{2 \mu-r_{o} v_{0}^{2}} \tag{3.42}
\end{equation*}
$$

Note that $2 \mu-r_{o} v_{0}^{2}<0$. Multiplying both sides by $-\left(2 \mu-r_{o} v_{0}^{2}\right)$ results in

$$
\begin{equation*}
r_{o} v_{0}^{2} d^{2} \geq r_{s}^{2} r_{o} v_{0}^{2}+2 r_{s}\left(r_{o}-r_{s}\right) \mu \tag{3.43}
\end{equation*}
$$

Eq. (3.43) shows the relationship of $d, v_{0}$ and $\mu$ for an avoidable collision. Solving Eq. (3.43) for $\mu$, and utilizing the definition of $\mu$ in Eq. (3.26), yield the maximum required charge criterion to avoid a collision with a given initial approach speed $v_{0}$ and miss-distance $d$.

$$
\begin{equation*}
Q_{\max } \geq \frac{m_{1} m_{2}}{m_{1}+m_{2}} \frac{r_{o} v_{0}^{2}\left(r_{s}^{2}-d^{2}\right)}{2 k_{\mathrm{c}} r_{s}\left(r_{o}-r_{s}\right)} \tag{3.44}
\end{equation*}
$$

For example, a large value of $\dot{r}\left(t_{0}\right)^{2}$ means SC 2 is approaching SC 1 at a high speed. Here $v_{0}$ is large, and according to Eq. (3.44), a large $Q_{\max }$ is required to avoid the collision. If the upper limit of the initial separation distance rate $\dot{r}\left(t_{0}\right)$ is known, then Eq. (3.44) tells us the minimum value of the saturated charge product needed to avoid the collision. For a given formation flying mission where the maximum magnitude of the possible separation distance rate has been determined, Eq. (3.44) helps us design the electric charge devices of the Coulomb-forced spacecraft to provide the maximum required repulsive forces.

Alternatively, solving Eq. (3.43) for $v_{0}$ yields the criterion for the magnitude of the relative velocity:

$$
\begin{equation*}
v_{0} \leq \sqrt{\frac{2 \mu r_{s}\left(r_{o}-r_{s}\right)}{r_{o}\left(d^{2}-r_{s}^{2}\right)}} \tag{3.45}
\end{equation*}
$$



Figure 3.5: Critical surface of parameters for an avoidable collision.

If the parameter $\mu$ of the spacecraft is given (specifically maximum spacecraft charge), then Eq. (3.45) tells us the maximum allowable relative velocity that guarantees the collision to be avoidable. As expected, the smaller the allowable charge levels, the smaller the allowable approach speeds $v_{0}$ are.

To provide insight into the relationship between the maximum charge and initial velocity, Figure 3.5 shows the critical surface of parameters $d, v_{0}$ and $Q_{\max }$ under the following conditions:

$$
\left\{\begin{array}{l}
m_{1}=50 \mathrm{~kg}  \tag{3.46}\\
m_{2}=50 \mathrm{~kg}
\end{array}, \quad\left\{\begin{array}{l}
r_{s}=4 \mathrm{~m} \\
r_{o}=18 \mathrm{~m}
\end{array}\right.\right.
$$

Parameters $d, v_{0}$ and $Q_{\max }$ in the region above the critical surface represent avoidable collisions. Beneath the surface are parameters of unavoidable collisions.

This critical surface is one quarter of a saddle surface. When the magnitude of the relative velocity $v_{0}$ is set, a larger the offset distance $d$ is, a smaller $Q_{\text {max }}$ is required. And when $d=r_{s}, Q_{\max }=0$, the trajectory of SC 2 will touch the safe region of SC1 $\mathcal{B}_{r_{s}}$ but won't penetrate it without any control. If the offset distance $d$ is set, a larger $v_{0}$ results in the bigger $\dot{r}_{0}$ component, thus a larger $Q_{\max }$ is required for a collision avoidance maneuver. When $v_{0}=0$, which means the two spacecraft are stationary to
each other, nothing needs to be done to avoid a collision, so $Q_{\max }$ in this case remains zero.

When $h=0$, then the offset distance $d=0$ and the spacecraft are lined up for a head-on collision. For this worst case situation, the criteria in Eq. (3.44) and Eq. (3.45) reduce to

$$
\begin{align*}
& Q_{\max } \geq \frac{\dot{r}\left(t_{0}\right)^{2}}{2 k_{\mathrm{c}}} \frac{m_{1} m_{2}}{m_{1}+m_{2}} \frac{r_{o} r_{s}}{r_{o}-r_{s}},  \tag{3.47}\\
& \dot{r}\left(t_{0}\right)^{2} \leq 2 \mu\left(\frac{1}{r_{o}}-\frac{1}{r_{s}}\right) . \tag{3.48}
\end{align*}
$$

Note that even though the derivation of the criteria is based on the assumption that $Q=Q_{\max }>0$ and $d<r_{s}$, the same procedure can also be performed in the case $Q=Q_{\text {min }}<0$ and $d>r_{s}$. In this case the two spacecraft are attracting each other. The problem is then changed to analyzing the requirements to prevent the two attracting spacecraft from colliding. Following the same procedure in deriving the criterion in Eq. (3.44), yields

$$
\begin{equation*}
Q_{\min } \geq \frac{m_{1} m_{2}}{m_{1}+m_{2}} \frac{r_{o} v_{0}^{2}\left(r_{s}^{2}-d^{2}\right)}{2 k_{\mathrm{c}} r_{s}\left(r_{o}-r_{s}\right)} \stackrel{\text { def }}{=} g . \tag{3.49}
\end{equation*}
$$

Eq. (3.49) has exactly the same form as Eq. (3.44). Because $d>r_{s}$, here $g<0$. It's assumed that the two spacecraft are attracting each other, so the charge product $Q$ is always negative. The smaller $Q$ is, the larger the attracting force becomes, and thus the more likely the two spacecraft will get closer. If in a mission the two spacecraft are fully charged such that $Q=Q_{\text {min }}$, then Eq. (3.49) tells us the minimum allowable value of the limit of the negative charge product $Q$, guaranteeing that the spacecraft won't collide.

### 3.5 Numerical Simulations

While the charge control is derived for the general 3-dimensional spacecraft motion, the conservation of angular momentum forces all resulting motion to be planar.

Thus, without loss of generality, the following numerical simulations all consider planar motion to simplify the visualizations.

The masses of the two spacecraft are $m_{1}=m_{2}=50 \mathrm{~kg}$. At first let us assume that the spacecraft are flying in deep space with the Debye length being $\lambda_{\mathrm{d}}=50 \mathrm{~m}$. The radii of the safe region $r_{s}$ and the potential region $r_{o}$ are determined by the requirements of a specific formation mission. For these simulations $r_{s}$ and $r_{o}$ are set as

$$
r_{s}=3 \mathrm{~m}, \quad r_{o}=16 \mathrm{~m} .
$$

The region of the effective control range $r_{c}$ will be given in specific simulation examples. The initial inertial coordinates and inertial velocities are

$$
\left\{\begin{array}{l}
\boldsymbol{R}_{1}=[-8,-3]^{T} \mathrm{~m}  \tag{3.50}\\
\boldsymbol{R}_{2}=[8,3]^{T} \mathrm{~m}
\end{array}, \quad\left\{\begin{array}{l}
\dot{\boldsymbol{R}}_{1}=[0.0060 .002]^{T} \mathrm{~m} / \mathrm{s} \\
\dot{\boldsymbol{R}}_{2}=[-0.006-0.002]^{T} \mathrm{~m} / \mathrm{s}
\end{array}\right.\right.
$$

These initial conditions are set up such that the spacecraft cluster's center of mass is stationary.

### 3.5.1 Simulation Without Control Truncation Or Charge Saturations

The unsaturated charge control law in Eq. (6.32) is guaranteed to prevent any collision. As to the coefficients of the controller, the larger $k_{1}$ is, the more the spacecraft proximity near $r_{s}$ is penalized. A larger $k_{2}$ results in more control effort in driving $\dot{r} \rightarrow-\dot{r}\left(t_{0}\right)$. For the first simulation the controller coefficients are chosen as

$$
\begin{equation*}
k_{1}=0.000001 \mathrm{kgm}^{4} \mathrm{~s} / \mathrm{C}^{2}, \quad k_{2}=0.0002 \mathrm{~s} / \mathrm{C} \tag{3.51}
\end{equation*}
$$

These coefficients result in a case where the state $x_{2}$ crosses zero and then converge to $b(r)$. Figure 3.6 shows the numerical simulation results with the initial conditions listed above. Note that here the effective control range $r_{c}$ is set to be infinity and no control truncation is occurring. The SC1 and SC2 start from a separation distance slightly larger than $r_{o}$. Before $r(t)=r_{o}$, the control is not triggered and the charges remain


Figure 3.6: Simulation results without truncation and charge saturations, in the case that $x_{2}$ crosses zero.
zeros. When $r(t)=r_{o}$, the control is triggered and the spacecraft start to repel each other. After about 1.3 hours, it is found that $r(t)>r_{o}$, and the control law is now only trying to equalize the radial separation rate magnitude to the initial value. The collision has already been avoided at this time. The following discussion illustrates the analytical predictions of the behaviors of $x_{2}$.

From Theorem $1 x_{2}$ converges to the interval $[0, b(r)]$. Further, it eventually converges to $b(r)$ if it crosses zero. Figure $3.6(\mathrm{~d})$ and $3.6(\mathrm{e})$ show the histories of the separation distance rate for different time spans. After $x_{2}$ crosses zero it keeps rising up as predicted. Figure 3.6(e) shows that $x_{2}$ crosses $b(r)$ at the point A. At this critical point $\dot{V}=0$ and $x_{2}=b(r)>0$. Using Eq. (6.32) the charge product can be solved as:

$$
\begin{equation*}
Q=-\frac{\left\|\boldsymbol{H}_{c}\right\|^{2}}{k_{\mathrm{c}} r}\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right) e^{\frac{r}{\lambda_{\mathrm{d}}}} \tag{3.52}
\end{equation*}
$$

From the separation distance equation of motion in Eq. (3.11), we find that the acceleration of the separation distance is $\ddot{r}=0$. Note that $\dot{x}_{2}=\ddot{r}$, thus $x_{2}$ stops increasing at point A, and starts to decrease. At point A, $\dot{x}_{2}=0$, and $x_{2}$ is bounded by $b$ function value at point A. So $x_{2}$ crosses the history of $b(r)$ because $b(r)$ is decreasing. After $x_{2}$ hits $b(r)$, it converges to the trajectory of $b(r)$ asymptotically because $\dot{V}<0$.

Figure 3.6(f) shows that after 15 hours the spacecraft start to attract each other to make $x_{2}$ to converge to $b(r)$. As shown in Figure 3.6(e), this is when the state $x_{2}$ becomes positive. Physically $x_{2}>0$ means that the separation rate is now larger than the original radial approach rate magnitude. To slow down the radial motion, the signs of the charges become opposite to yield attractive forces. The reason the magnitudes of the charges are increasing here is that the separation distance has already grown very large. Even though the required control force is very small, the $1 / r^{2}$ dependency of the Coulomb force expression requires a large spacecraft charge to generate it. This issue has little to no practical consequence because the collision avoidance maneuver was effectively finished after about 1.3 hours. This long term behavior is illustrated to


Figure 3.7: Simulation results without truncation and charge saturations, in the case that $x_{2}$ won't cross zero.
provide a numerical example of the analytically predicted behaviors of $x_{2}$.
When the controller's coefficients are set to

$$
\begin{equation*}
k_{1}=0.0002 \mathrm{kgm}^{4} \mathrm{~s} / \mathrm{C}^{2}, \quad k_{2}=0.0001 \mathrm{~s} / \mathrm{C} . \tag{3.53}
\end{equation*}
$$

With these parameters the state $x_{2}$ does not reach zero, as seen in the simulation results in Figure 3.7. But $x_{2}$ still converges asymptotically to zero from a negative value. As shown in Figure 3.7(d), because $k_{1}$ is large, the control charges that penalizes the spacecraft proximity near $r_{s}$ dominate in the initial one hour. The first peak of the charge product happens when the two spacecraft get closest. Physically, when the craft get close, the repulsive force suddenly increases to a peak to repel the craft. This results
in a sharp trajectory of the spacecraft as shown in Figure 3.7(a).

### 3.5.2 Simulation With Charge Truncations

In the following simulations the control is truncated when the separation distance is larger than $r_{c}$ where $r_{c}>r_{o}$. The range of the control is denoted by $\mathcal{B}_{r_{c}}$ with radius $r_{c}$. The control charges are turned off when the separation distance $r(t)>r_{c}$. Setting $r_{c}=20 \mathrm{~m}$, and the controller's coeficients

$$
\begin{equation*}
k_{1}=0.0001 \mathrm{kgm}^{4} \mathrm{~s} / \mathrm{C}^{2}, \quad k_{2}=0.0003 \mathrm{~s} / \mathrm{C} \tag{3.54}
\end{equation*}
$$

and using the previous spacecraft initial position and velocity conditions, yields Figure 3.8 that shows the simulation results in this case.

Because of the charge truncations, $x_{2}$ is not guaranteed to converge to zero during this maneuver. However, as the control analysis predicts, the radial rate tracking error $x_{2}$ will remain bounded while achieving a collision avoidance maneuver where $x_{1} \rightarrow 0$.

To test the robustness of the control, the spacecraft are put in an geostationary orbit to compare the performance with that of the spacecraft flying in deep space. The initial conditions in Eq. (3.50) are treated as LVLH frame position and velocity vectors, which are then mapped into inertial vectors with respect to the Earth centered inertial frame. The full nonlinear equations of motion are then integrated with the same charge collision avoidance control applied. After the integration the resulting motion is mapped back into equivalent LVLH frame position vectors, where the rotating LVLH frame is assumed to be the spacecraft cluster's center of mass. The simulation results are illustrated in Figure 3.8 simultaneously with the simulation performed in deep space.

The parameters of the two spacecraft and the controller are kept unchanged to the truncated control example. In GEO the Debye length ranges from 100-1000 meters. But for a fair comparison, here the Debye length is still set to be $\lambda_{d}=50 \mathrm{~m}$. While the trajectories in Figure 3.8(a) are different for deep space and GEO cases, they both


Figure 3.8: Simulation results with truncation but without charge saturations.
yield a separation distance $r(t)$ that is always greater than the safety limit $r_{s}$. From the charge control law in Eq. (6.32), the charge product $Q$ will increase if $r(t)$ gets too close to $r_{s}$. In fact, $Q \rightarrow \infty$ if $r(t) \rightarrow r_{s}$. Thus while the orbital motion is not analyzed explicitly in this study, if the collision avoidance happens quickly enough as compared to the orbital dynamics, the algorithm can still be effective.

### 3.5.3 Simulation With Charge Saturations

When the spacecraft charge saturations are introduced, a potential collision is unpreventable if the two spacecraft are flying towards each other at a very high speed. Eq. (3.44) and (3.45) provide the criteria for an avoidable potential collision. Note that
even though the analysis of the criteria is based on the assumption that the Debye length $\lambda_{d} \rightarrow \infty$, the following numerical simulation still has the Debye length set to $\lambda_{d}=50 \mathrm{~m}$ to show how close the simplified charge limit estimation in Eq. (3.44) is with that of a more complex motion with a limited Debye length. With the same initial conditions as in the previous simulation examples in Eq. (3.50), the initial offset distance $d$ and the magnitude of initial relative velocity $v_{0}$ are

$$
d=0.6325 \mathrm{~m}, \quad v_{0}=0.0126 \mathrm{~m} / \mathrm{s}
$$

Utilizing $r_{s}, r_{o}$ and masses $m_{i}$, the critical charge product for an avoidable collision is

$$
\begin{equation*}
Q_{\mathrm{C}}=\frac{m_{1} m_{2}}{m_{1}+m_{2}} \frac{r_{o} v_{0}^{2}\left(r_{s}^{2}-d^{2}\right)}{2 k_{\mathrm{c}} r_{s}\left(r_{o}-r_{s}\right)}=7.8492 \times 10^{-13} \mathrm{C}^{2} \tag{3.55}
\end{equation*}
$$

The critical saturation limit for each individual charge is $q_{c}=\sqrt{Q_{\mathrm{C}}}=0.88596 \mu \mathrm{C}$.
Figure 3.9 shows simulation results with the same initial conditions but different charge saturation limits. It is assumed that in the potential region $\mathcal{B}_{r_{o}}$ the two spacecraft are fully charged to repel each other. This can be achieved by setting the controller's coefficients $k_{1}$ and $k_{2}$ to be some large numbers. Here the controller's parameters are set to be

$$
\begin{equation*}
k_{1}=0.1 \mathrm{kgm}^{4} \mathrm{~s} / \mathrm{C}^{2}, \quad k_{2}=0.1 \mathrm{~s} / \mathrm{C} \tag{3.56}
\end{equation*}
$$

It can be seen that a larger $q_{\max }$ results in a more aggressive repulsion with a larger periapses radius. When $q_{\max }=q_{c}$, the closest distance is slightly smaller than $r_{s}$. SC 2 penetrates about 0.21 m inside the safe restraint region $\mathcal{B}_{r_{s}}$ with $r_{s}=3 \mathrm{~m}$. This happens because the Debye length effect partially shields the electrostatic force between the spacecraft. Note that in real space missions, $r_{s}$ is a safety-restraint distance estimate that guarantees no physical contact happens and the electrical devices on both spacecraft won't interfere with each other. The 0.25 m 's penetration is not large when compared with $r_{s}$, only $7 \%$. On the other hand, if we know how much Debye shielding will occur, the value of $r_{s}$ can be adjusted to be larger such that the closest distance between the


Figure 3.9: Simulation results with charge saturation.
spacecraft is still big enough to keep the spacecraft and all the devices safe. At this point, it can be concluded that the estimation of the charge product criterion in Eq. (3.44) is sufficiently accurate to provide a practical maximum required charge computation.

### 3.6 Conclusion

A Coulomb-force based collision avoidance control problem of two spacecraft is discussed. After formulating the equation of motion of the separation distance, a collision avoidance charge control law with the feedback of the separation distance and the distance rate is developed based on Lyapunov's direct method. Without saturation and truncation of the spacecraft charges, the control is able to prevent collisions while keep-
ing the final kinetic energy the same as the initial kinetic energy. The charge truncation introduces an uncertainty in maintaining the relative kinetic energy, but the collision avoidance purpose is still achieved and the change in kinetic energy is guaranteed to be bounded. The charge saturation may lead to a failure in achieving a collision avoidance. Analytical conditions for a preventable collision are formulated by ignoring the plasma shielding effect. Simulations show that the predicted minimum separation distance obtained using the analytical criteria is close to the actual minimum distance when the plasma shielding effect is taken into account, thus the criteria are practically usable.

## CHAPTER 4

## OPEN-LOOP ELECTROSTATIC SPACECRAFT COLLISION AVOIDANCE USING PIECE-WISE CONSTANT CHARGES

Chapter 3 develops a Lyapunov-based control strategy to make a collision avoidance maneuver between two spacecraft. That control strategy achieves two objectives: avoid the collision and keep the final departure speed magnitude bounded to the iniital approaching speed magnitude. Without charge saturations the controller can prevent any collision. Considering charge saturations, the chapter finds the analytical criteria for an avoidable collision are determined by assuming the Debye length to be infinity. While this feedback control strategy can maintain specified safety separation distances, this control will cause the craft to depart in a different direction from when the collision avoidance maneuver started.

This chapter investigates a Coulomb force control strategy to achieve one more objective: make the final speed in the same direction as the initial speed. A new openloop control approach is presented. By assuming the Debye length to be large compared to the separation distance, and that the spacecraft charges are piece-wise constant, the relative EOM has exactly the same form as gravitational two body problem (G2BP). Thus the relative trajectory of the spacecraft is a conic section [12]. Through switching the values of the spacecraft charges, a patched conic section trajectory is investigated which will satisfy the separation distance and the departure velocity requirements. Of interest are the charge magnitudes of these maneuvers. Numerical simulation will illus-
trate how these open-loop charge solutions will generate the desired collision avoidance maneuver. The work in this chapter has been presented as conference papers in References $[25,26]$ and has been published as a journal article in Reference [27].


Figure 4.1: Illustration of the 2 -spacecraft system.

### 4.1 Charged Spacecraft Equations of Motion

Consider two spacecraft free-flying in 3-dimensional space where there are no external forces acting on the system. The scenario of the two body system is shown in Figure 4.1. Assuming point-charge models for the spacecraft, the inertial equations of motion of the two charged spacecraft are

$$
\begin{align*}
& m_{1} \ddot{\boldsymbol{R}}_{1}=-k_{\mathrm{c}} \frac{q_{1} q_{2}}{r^{2}}\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right) e^{-\frac{r}{\lambda_{\mathrm{d}}}} \hat{\boldsymbol{e}}_{r}  \tag{4.1a}\\
& m_{2} \ddot{\boldsymbol{R}}_{2}=k_{\mathrm{c}} \frac{q_{1} q_{2}}{r^{2}}\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right) e^{-\frac{r}{\lambda_{\mathrm{d}}} \hat{\boldsymbol{e}}_{r}} \tag{4.1b}
\end{align*}
$$

The notations of variables are the same as in Chapter 3. For the reader's convenience, the notations are briefly repeated here. The parameter $k_{\mathrm{c}}=8.99 \times 10^{9} \mathrm{C}^{-2} \cdot \mathrm{~N} \cdot \mathrm{~m}^{2}$ is the Coulomb constant, $r$ is the separation distance between the two spacecraft, $\boldsymbol{r}$ is the relative position vector pointing from spacecraft 1 (SC1) to spacecraft $2(\mathrm{SC} 2)$, $\hat{\boldsymbol{e}}_{r}$ is the unit vector of $\boldsymbol{r}$, and $\lambda_{\mathrm{d}}$ is the Debye length. $\boldsymbol{R}_{i}$ is the inertial position vector of
the $i^{\text {th }}$ spacecraft. The inertial relative acceleration vector $\ddot{\boldsymbol{r}}$ is

$$
\begin{equation*}
\ddot{\boldsymbol{r}}=\ddot{\boldsymbol{R}}_{2}-\ddot{\boldsymbol{R}}_{1}=\frac{k_{\mathrm{c}} q_{1} q_{2}}{m_{1} m_{2} r^{2}}\left(m_{1}+m_{2}\right)\left(1+\frac{r}{\lambda_{\mathrm{d}}}\right) e^{-\frac{r}{\lambda_{\mathrm{d}}}} \hat{\boldsymbol{e}}_{r} \tag{4.2}
\end{equation*}
$$

Note that these equations do not explicitly consider planetary gravity acting on the spacecraft. However, if the collision avoidance maneuver time is very small compared to the cluster orbital period, then they can also be considered an approximation of the charged relative orbital motion. For example, a GEO spacecraft collision avoidance maneuver which takes minutes would be very short compared to the 1 day orbit period, and thus the relative orbital motion would have a secondary effect on the relative motion.

This chapter is going to find a symmetric patched conic section trajectory to prevent a collision, while forcing the departure velocity vector to be the same as the initial arrival velocity vector. Reference 12 shows that if $\lambda_{d} \rightarrow \infty$, and the charge product $Q=q_{1} q_{2}$ is constant, then the relative motion trajectory of the two spacecraft is a conic section. Letting $\lambda \rightarrow \infty$ and defining

$$
\begin{equation*}
\mu=-k_{\mathrm{c}} \frac{Q\left(m_{1}+m_{2}\right)}{m_{1} m_{2}} \tag{4.3}
\end{equation*}
$$

Eq. (4.2) is rewritten as

$$
\begin{equation*}
\ddot{r}=-\frac{\mu}{r^{3}} \boldsymbol{r} \tag{4.4}
\end{equation*}
$$

Eq. (4.4) has exactly the same algebraic form as the EOM of G2BP. If the charge product $Q$ is constant, then the effective gravitational coefficient $\mu$ is also constant. Thus the resulting motion can be described by a conic section. Note that here $\mu$ can be positive or negative. For the oppostive charge sign case $Q<0$, resulting in a positive effective gravitational constant $\mu>0$. In this case Eq. (4.4) is exactly the same as the G2BP. If $Q>0$ and $\mu<0$, then the relative trajectory is a repulsive hyperbola, where SC 2 is moving along a hyperbola, and SC1 stays at the farther focus point. [12]


Figure 4.2: Illustration of the symmetric patched conic section relative motion trajectory with respect to mass $m_{1}$.

### 4.2 3-Phase Symmetric Trajectory Scenario

For a Coulomb-forced two spacecraft collision avoidance problem, generally there are an infinity of possible charge and charge switching time solutions which achieve a collision avoidance. This chapter investigates symmetric trajectory programming approach to avoid a collision as well as hold the relative velocity.

An example of the 3-Phase symmetric relative trajectory scenario is shown in Figure 4.2. At the beginning, the two spacecraft are flying freely and approaching each other such that their minimum separation distance will violate a desired safety distance $r_{s}$. At the point $A$, the separation distance $r$ between the spacecraft reaches a potential collision region range $r_{o}$. The spacecraft are charged such that $Q>0$ and the spacecraft start to repel each other to avoid the collision. The magnitude of the charge product is held constant in Phase I until point $B$ is reached. Thus the trajectory $\widehat{A B}$ is a repulsive hyperbola. At point $B$ the charge product switches to a negative value such that the spacecraft are attracting each other. During Phase II from the point $B$ to the point $C$, the charge product is again held constant. The arc $\widehat{B C}$ is an attractive conic section which can be ellipsis, parabola, or hyperbola depending on the relative arrival velocity magnitude. At the point $C$ the charge product switches back to the same value as in arc $\widehat{A B}$ to produce a symmetric trajectory to $\widehat{A B}$. At the point $E$, the charges are turned off and the spacecraft begin to fly freely in space. The entire trajectory is symmetric
about the axis $\overline{O D}$. And the axis $\overline{O D}$ is the line crossing SC 1 and perpendicular to the initial relative velocity.


Figure 4.3: Scenario of the circular Phase II trajectory.

### 4.3 Circular Transitional Orbit Programming

Before studying the general symmetric trajectories, let us first investigate a special case where the Phase II trajectory is a section of a circle as illustrated in Figure 4.3. Assume that the relative position vector $\boldsymbol{r}_{A}$ and the relative velocity $\dot{\boldsymbol{r}}_{A}$ at point A can be measured. From the description of the symmetric trajectory scenario in the last section there are five unknowns that need to be determined: three charge products $Q_{\mathrm{I}}$, $Q_{\mathrm{II}}$ and $Q_{\mathrm{III}}$, and two charge switching times at points $B$ and $C$. To solve for these five variables some constraints need be clarified.

### 4.3.1 Constraints

For Phase I $\widehat{A B}$ and Phase III $\widehat{C E}$ to be symmetric, the charge products should be the same value. Thus the first constraint is

$$
\begin{equation*}
Q_{\mathrm{III}}=Q_{\mathrm{I}} \tag{4.5}
\end{equation*}
$$

Because the trajectory of Phase II $\widehat{B C}$ is a section of a circle, its shape is always symmetric about the symmetry axis $\overline{O D}$. Thus a symmetric arc $\widehat{B C}$ requires only that
the angle $\angle D O C$ satisfies

$$
\begin{equation*}
\angle D O C=\angle B O D \tag{4.6}
\end{equation*}
$$

The point $B$ connects Phase I and Phase II. Thus $\dot{\boldsymbol{r}}_{B}$ must be perpendicular to $\boldsymbol{r}_{B}$, which means the point $B$ is the periapsis of Phase I . This results in the third constraint:

$$
\begin{equation*}
r_{B}=r_{p \mathrm{I}} \tag{4.7}
\end{equation*}
$$

The trajectory of Phase II is a section of circle, this requirement can be formulated using the angular momentum magnitude

$$
\begin{equation*}
h_{\mathrm{II}}^{2}=\mu_{\mathrm{II}} r_{B} \tag{4.8}
\end{equation*}
$$

The collision avoidance task requires that the separation distance $r(t)$ must be greater than a certain safe-restraint distance $r_{s}$ for all time:

$$
\begin{equation*}
r(t) \geq r_{s} \tag{4.9}
\end{equation*}
$$

This constraint is global and comes from the collision avoidance mission. For the convenience of calculation, this safety constraint is expressed by the condition

$$
\begin{equation*}
r_{\min }=\gamma r_{s} \tag{4.10}
\end{equation*}
$$

where $\gamma \geq 1$. In the case that Phase II is a section of a circle, $r_{\min }=r_{B}$. Thus the final safety constraint for a circular transitional symmetric trajectory is

$$
\begin{equation*}
r_{B}=\gamma r_{s} \tag{4.11}
\end{equation*}
$$

Now five constraints in Eqs. (4.5)-(4.8), and (4.11) have been found.

### 4.3.2 Circular Transitional Orbit Algorithm

The symmetric constraint in Eq. (4.5) provides $Q_{\mathrm{III}}$ once $Q_{\mathrm{I}}$ is obtained. Note that the angle $\angle D O C$ is the true anomaly angle (in case the point $D$ is the periapsis)
of Phase II. Once the conic section properties of Phase II, especially $Q_{\mathrm{II}}$, is achieved, the charge switching time at the point $C$ can be determined by using Kepler's equation and the symmetry constraint in Eq. (4.6).

Now three variables ( $Q_{\mathrm{I}}, Q_{\mathrm{II}}, t_{\mathrm{I}}$ ) are left to be determined. The conic section properties of Phase I are solved using $\boldsymbol{r}_{A}$ and $\dot{\boldsymbol{r}}_{A}$. The eccentricity vector of Phase I is

$$
\begin{equation*}
\boldsymbol{c}_{\mathrm{I}}=\dot{\boldsymbol{r}}_{A} \times \boldsymbol{h}-\frac{\mu_{\mathrm{I}}}{r_{A}} \boldsymbol{r}_{A} \tag{4.12}
\end{equation*}
$$

where $\boldsymbol{h}=\boldsymbol{r}_{A} \times \dot{\boldsymbol{r}}_{A}$ is the specific angular momentum of the system, and

$$
\begin{equation*}
\mu_{\mathrm{I}}=-k_{\mathrm{c}} \frac{Q_{\mathrm{I}}\left(m_{1}+m_{2}\right)}{m_{1} m_{2}} \tag{4.13}
\end{equation*}
$$

is the effective gravitational coefficient of Phase I. Actually, by Eq. (4.3), finding the charge products $Q_{\mathrm{I}}$ and $Q_{\mathrm{II}}$ is equivalent to finding $\mu_{\mathrm{I}}$ and $\mu_{\mathrm{II}}$. Further, note the notation where $r_{A}=\left|\boldsymbol{r}_{A}\right|$. The vector $\boldsymbol{h}$ is constant by the assumption that there are no external forces acting on the sytem. The eccentricity and semi-major axis of Phase I are calculated by

$$
\begin{align*}
& e_{\mathrm{I}}=-\frac{\left\|c_{\mathrm{I}}\right\|}{\mu_{\mathrm{I}}}  \tag{4.14a}\\
& a_{\mathrm{I}}=\frac{r_{A} \mu_{\mathrm{I}}}{2 \mu_{\mathrm{I}}-r_{A} v_{A}^{2}} \tag{4.14b}
\end{align*}
$$

where $v_{A}=\left\|\dot{\boldsymbol{r}}_{A}\right\|$ is the magnitude of the relative velocity vector. The angle $\angle A O D$ is calculated as

$$
\begin{equation*}
\angle A O D=\arctan \left(\frac{h}{r_{A} v_{A}}\right)-\frac{\pi}{2} \tag{4.15}
\end{equation*}
$$

By applying the constraint that the point B must be the periapsis of Phase I, the charge switching time $t_{B}$ at point $B$ is calculated through

$$
\begin{equation*}
t_{B}=\frac{\left|N_{A \mathrm{I}}\right|}{\sqrt{\mu_{\mathrm{I}} / a_{\mathrm{I}}}} \tag{4.16}
\end{equation*}
$$

with the right hand side of this equation being completely determined by $\mu_{\mathrm{I}}$, which in return is determined by $Q_{\mathrm{I}}$. Thus, it can be concluded that the Phase I trajectory are determined by the charge product $Q_{\mathrm{I}}$.

The radius $r_{B}$ is calculated by

$$
\begin{equation*}
r_{B}=\frac{h^{2} / \mu_{\mathrm{I}}}{1-e_{\mathrm{I}}} \tag{4.17}
\end{equation*}
$$

where the eccentricity $e_{\mathrm{I}}$ is given by Eq. (4.14a). Substituting Eq. (4.17) into the safety constraint in Eq. (4.11) and multiplying both sides by $\mu_{\mathrm{I}}\left(1-e_{\mathrm{I}}\right) /\left(\gamma r_{s}\right)$, yield

$$
\begin{equation*}
\mu_{\mathrm{I}}\left(1-e_{\mathrm{I}}\right)=\frac{h^{2}}{\gamma r_{s}} \tag{4.18}
\end{equation*}
$$

Subtracting both sides by $\mu_{\mathrm{I}}$, taking square of both sides and using $e_{\mathrm{I}}=-\frac{\|\boldsymbol{c}\|}{\mu_{\mathrm{I}}}=$ $\left\|\frac{\boldsymbol{r}_{A}}{r_{A}}-\frac{\dot{\boldsymbol{r}}_{A} \times \boldsymbol{h}}{\mu_{\mathrm{I}}}\right\|$, yield

$$
\begin{equation*}
\mu_{\mathrm{I}}^{2} e_{\mathrm{I}}^{2}=\mu_{\mathrm{I}}^{2}-2 \mu_{\mathrm{I}} \dot{\boldsymbol{r}}_{A} \times \boldsymbol{h} \cdot \boldsymbol{r}_{A} / r_{A}+\dot{\boldsymbol{r}}_{A} \times \boldsymbol{h} \cdot \dot{\boldsymbol{r}}_{A} \times \boldsymbol{h} \tag{4.19}
\end{equation*}
$$

Substituting Eq. (4.19) into Eq. (4.18), yields

$$
\begin{equation*}
-2 \mu_{\mathrm{I}} \dot{\boldsymbol{r}}_{A} \times \boldsymbol{h} \cdot \boldsymbol{r}_{A} / r_{A}+\dot{\boldsymbol{r}}_{A} \times \boldsymbol{h} \cdot \dot{\boldsymbol{r}}_{A} \times \boldsymbol{h}=\frac{h^{4}}{\gamma^{2} r_{s}^{2}}-\frac{2 \mu_{\mathrm{I}} h^{2}}{\gamma r_{s}} \tag{4.20}
\end{equation*}
$$

Thus the Phase I effective gravitational coefficient for a circular transitional trajectory is solved by grouping terms containing $\mu_{\mathrm{I}}$ :

$$
\begin{equation*}
\mu_{\mathrm{I}, \mathrm{c}}=\frac{1}{2} \frac{\frac{h^{4}}{\gamma^{2} r_{s}^{2}}-\dot{\boldsymbol{r}}_{A} \times \boldsymbol{h} \cdot \dot{\boldsymbol{r}}_{A} \times \boldsymbol{h}}{\frac{h^{2}}{\gamma r_{s}}-\dot{\boldsymbol{r}}_{A} \times \boldsymbol{h} \cdot \frac{\boldsymbol{r}_{A}}{r_{A}}} \tag{4.21}
\end{equation*}
$$

After obtaining $\mu_{\mathrm{I}, \mathrm{c}}$, the variable $t_{\mathrm{I}}$ is determined by Eq. (4.16). These values of $\mu_{\mathrm{I}}$ and $t_{\mathrm{I}}$ ensure that at the point B the relative speed vector is perpendicular to the relative position vector, meanwhile the safety constraint $r_{B}=\gamma r_{s}$ is also satisfied.

The next step is to find a proper $Q_{\text {II }}$ or $\mu_{\mathrm{II}}$ that results in a circular orbit. Using the constraint for a circular transitional orbit in Eq. (4.8), $\mu_{\mathrm{II}}$ is found to be

$$
\begin{equation*}
\mu_{\mathrm{II}, \mathrm{c}}=\frac{h^{2}}{r_{B}}=\frac{h^{2}}{\gamma r_{s}} \tag{4.22}
\end{equation*}
$$

To find the Phase II duration time $t_{\mathrm{II}}$, the Phase II symmetry constraint in Eq. (4.6) is utilized. Note that the angular velocity is constant in Phase II, the duration time is proportional to the angle $\angle B O C$ as:

$$
\begin{equation*}
t_{\mathrm{II}, \mathrm{c}}=\angle B O C \cdot \frac{T_{\mathrm{II}}}{2 \pi}=2 \angle B O D \cdot \frac{T_{\mathrm{II}}}{2 \pi}=\frac{\angle B O D \cdot T_{\mathrm{II}}}{\pi} \tag{4.23}
\end{equation*}
$$

The period of the Phase II circular orbit is $T_{\mathrm{II}}=\sqrt{\mu_{\mathrm{II}} / r_{\mathrm{II}}^{3}}$, while the angle $\angle B O D$ is given by

$$
\begin{equation*}
\angle B O D=\angle A O D-\left|f_{A I}\right|=\angle A O D+\operatorname{atan}\left(\frac{\hat{\boldsymbol{i}}_{c \mathrm{I}} \times \hat{\boldsymbol{i}}_{r A} \cdot \hat{\boldsymbol{i}}_{h}}{\hat{\boldsymbol{i}}_{\mathrm{cI}} \cdot \hat{\boldsymbol{i}}_{r A}}\right) \tag{4.24}
\end{equation*}
$$

where $\hat{\boldsymbol{i}}_{\text {cI }}, \hat{\boldsymbol{i}}_{r A}$ and $\hat{\boldsymbol{i}}_{h}$ are the unit vectors of $\boldsymbol{c}_{\mathrm{I}}, \boldsymbol{r}_{A}$ and $\boldsymbol{h}$ respectively. The angle $\angle A O D$ is expressed in Eq. (4.15).

Thus a symmetric trajectory with Phase II being a part of a circular orbit has been found. Specifically, the variables $\mu_{\mathrm{I}}, \mu_{\mathrm{II}}, Q_{\mathrm{III}}, t_{B}, t_{\mathrm{II}}$ are calculated through Eqs. (4.21), (4.22), (4.5), (4.16) and (4.23), respectively. Note that this circular transitional trajectory solution is calculated analytically.

### 4.4 General Symmetric Trajectory Programming Strategy

After solving a circular Phase II trajectory in the last section, this section is going to investigate the more general symmetric collision avoidance trajectories with the Phase II trajectory being any type of conic section.

A general 3-Phase symmetric trajectory is shown in Figure 4.2. As mentioned in the last section, as with the circular Phase II case, there are five unknowns that need to be determined: $\left[Q_{\mathrm{I}}, Q_{\mathrm{II}}, Q_{\mathrm{III}}, t_{B}, t_{\mathrm{II}}\right]$.

### 4.4.1 Constraints

The general constraints are largely the same as those for the circular transitional orbit. The three constraints in Eqs. (4.5), (4.6), and (4.10) are directly used to find a general symmetric trajectory. Because here the Phase II trajectory is a part of general conic section, the circular constraints in Eqs. (4.7) and (4.8) are not applicable.

Since the arc $\widehat{B C}$ is not a part of circle, for Phase II to be symmetric about the symmetric axis $\overline{O D}$, the point $D$ must be the periapsis or apoapsis of Phase II, unless
the $\operatorname{arc} \widehat{B C}$ is a part of a circular orbit. This requirement is formulated as:

$$
\begin{equation*}
r_{D}=r_{p, \mathrm{II}}, \quad \text { or } \quad r_{D}=r_{a, \mathrm{II}} \tag{4.25}
\end{equation*}
$$

where $r_{p, \text { II }}$ and $r_{a, \text { II }}$ are the periapsis radius and apoapsis radius of Phase II.
Now there are four equality constraints to solve the patched conic collision avoidance trajectory. Eqs. (4.5), (4.6) and (4.25) are from the symmetric patched conic section properties. These three constraints ensure a symmetric trajectory. The constraint given by Eq. (4.11) is required by the collision avoidance task. To complete the 5 -variable searching problem, one more constraint is needed.

Note that the four equality constraints ensure a collision avoidance and meanwhile result in a symmetric trajectory, which means the symmetric trajectory programming requirements have been satisfied. The remaining one degree of freedom actually provides a flexibility to search the five variables. Here this section assumes a proper value of $Q_{\mathrm{I}}$, then constructs a closed-loop numerical iteration routine to find other four variables. This iteration routine can be used as a part of the charge-optimal trajectory programming problem which updates $Q_{\mathrm{I}}$ such that a certain charge cost function is minimized.

### 4.4.2 General Numerical Iteration Routine

A numerical iteration routine is desired to find a symmetric patched conic section trajectory for the collision avoidance problem, assuming that a proper value of $Q_{\mathrm{I}}$ has been set. The charge product $Q_{\mathrm{I}}$ and the initial conditions $\left[\boldsymbol{r}_{A}, \dot{\boldsymbol{r}}_{A}\right]$ determine the conic section of Phase I. Without loss of generality, assume that $t_{A}=0$. If $t_{B}$ is given, the angle $\angle A O B$ can be calculated using Kepler's equation in Phase I. The states $\left[\boldsymbol{r}_{B}, \dot{r}_{B}\right]$ are also determined by solving the orbit EOM of Phase I. Utilizing the constraint that the point $D$ must be the periapsis or apoapsis of Phase II, the point $C$ is determined by the constraint in Eq. (4.6). Phase III is determined by the state of point $C$, which can
be infered from $t_{B}$. Thus the charge switching time variable $t_{B}$ logically determines the whole patched conic section trajectory. In the numerical iteration routine, $t_{B}$ is chosen as the variable to be propagated.

Because $t_{B}$ has been chosen as the variable to be propagated in the iteration loop, it starts from an initial guess value, and is updated using an error of a target function. In the present formulation of the algorithm the time point $t_{B}$ is assumed to be given. The states at point $B$ are determined by using the conic section properties of Phase I. The mean hyperbolic anomaly of point $B$ considered in Phase I is calculated using the Kepler's equation:

$$
\begin{equation*}
N_{B \mathrm{I}}=N_{A \mathrm{I}}+\sqrt{\frac{\mu_{\mathrm{I}}^{3}}{a_{\mathrm{I}}^{3}}} \cdot t_{B}=N_{A \mathrm{I}}+n_{\mathrm{I}} \cdot t_{B} \tag{4.26}
\end{equation*}
$$

Then the hyperbolic anomaly $H_{B I}$ is calculated by numerically solving the standard anomaly relationship: [28]

$$
\begin{equation*}
N_{B \mathrm{I}}=e_{\mathrm{I}} \sinh \left(H_{B \mathrm{I}}\right)+H_{B \mathrm{I}} \tag{4.27}
\end{equation*}
$$

Thus the true anomaly of point $B$ in Phase I is determined by

$$
\begin{equation*}
f_{B, \mathrm{I}}=2 \cdot \arctan \left(\tanh \left(\frac{H_{B \mathrm{I}}}{2}\right) \sqrt{\frac{e_{\mathrm{I}}+1}{e_{\mathrm{I}}-1}}\right) \tag{4.28}
\end{equation*}
$$

The radius and the magnitude of the relative velocity at point $B$ are

$$
\begin{align*}
& r_{B}=\frac{h^{2} / \mu_{\mathrm{I}}}{1-e_{\mathrm{I}} \cos f_{B \mathrm{I}}}  \tag{4.29a}\\
& v_{B}=\sqrt{\mu_{\mathrm{I}}\left(\frac{2}{r_{B}}-\frac{1}{a_{\mathrm{I}}}\right)} \tag{4.29b}
\end{align*}
$$

here $h$ is the magnitude of the specific angular momentum determined by initial conditions. Eq. (4.29b) is obtained from the energy equation.

After obtaining the relative motion states at point $B$, Phase II can be determined by the symmetric conic section constraints. Specifically, the charge product $Q_{\mathrm{II}}$ and point $C$ can be calculated. At first, the angle $\angle A O B$ is calculated by

$$
\begin{equation*}
\angle A O B=\left|f_{B, \mathrm{I}}-f_{A, \mathrm{I}}\right| \tag{4.30}
\end{equation*}
$$

The angle $\angle B O D$ is determined by the geometry relation:

$$
\begin{equation*}
\angle B O D=\angle A O D-\angle A O B \tag{4.31}
\end{equation*}
$$

According to the symmetric constraint in Eq. (4.6), the angle

$$
\begin{equation*}
\angle C O D=\angle B O D \tag{4.32}
\end{equation*}
$$

is determined. Thus the point $C$ is located. Note that of the five variables which determine the symmetric conic section trajectory, the points $B, C$, the charge products $Q_{\mathrm{I}}, Q_{\mathrm{III}}$ have been solved. The only variable left to be determined is the charge product $Q_{\mathrm{II}}$. From the definition of $\mu$ in Eq. (4.4) we find:

$$
\begin{equation*}
\mu_{\mathrm{II}}=-k_{\mathrm{c}} \frac{Q_{\mathrm{II}}\left(m_{1}+m_{2}\right)}{m_{1} m_{2}} \tag{4.33}
\end{equation*}
$$

Once $\mu_{\mathrm{II}}$ is solved, $Q_{\mathrm{II}}$ is also determined. The following development is going to solve for $\mu_{\mathrm{II}}$ based on the states of the point $B$ and the symmetric constraints.

Since the arc $\widehat{B C}$ is a part of a conic section, it has all of the properties of conic section orbit. Utilizing the vis-viva equation, the eccentricity $e$ is expressed as:

$$
\begin{equation*}
e=\sqrt{1+\left(\frac{v^{2}}{\mu}-\frac{2}{r}\right) \frac{h^{2}}{\mu}} \tag{4.34}
\end{equation*}
$$

For a given two body system without external forces, the specific angular momentum $h$ is constant. Thus the expression of the eccentricity in Eq. (4.34) contains only three variables $r, v$ and $\mu$. Substituting Eq. (4.34) into the radius equation, yields

$$
\begin{equation*}
r=\frac{h^{2}}{\mu+\cos f \sqrt{\mu^{2}+\left(v^{2}-\frac{2 \mu}{r}\right) h^{2}}} \tag{4.35}
\end{equation*}
$$

Transforming Eq. (4.35) to separate the square root term, yields

$$
\begin{equation*}
\cos f \sqrt{\mu^{2}+\left(v^{2}-\frac{2 \mu}{r}\right) h^{2}}=\frac{h^{2}}{r}-\mu \tag{4.36}
\end{equation*}
$$

Squaring Eq. (4.36) and using the fact that $1-\cos ^{2} f=\sin ^{2} f$, Eq. (4.36) can be simplified to

$$
\begin{equation*}
\sin ^{2} f \mu^{2}-\frac{2 h^{2}}{r} \sin ^{2} f \mu-\cos ^{2} f v^{2} h^{2}+\frac{h^{4}}{r^{2}}=0 \tag{4.37}
\end{equation*}
$$

With $h$ being constant, this quadratic equation of $\mu$ contains the variables $f, r$ and $v$. Note that Eq. (4.37) is valid for all conic section orbits. To solve for $\mu_{\mathrm{II}}$, one just need to evaluate $f, r$ and $v$ at point $B$ in Phase II and solve the quadratic equation in Eq. (4.37). $r_{B}$ and $v_{B}$ are given by Eq. (4.29). If the point $D$ is the periapsis location, then

$$
\begin{equation*}
f_{B, I I}=-\angle B O D \tag{4.38}
\end{equation*}
$$

Else, if $D$ is the apoapsis location, then

$$
\begin{equation*}
f_{B, \mathrm{II}}=\pi-\angle B O D \tag{4.39}
\end{equation*}
$$

In both cases, the resulting final equations after substituting $f_{B I I}$ into Eq. (4.37) are identical:

$$
\begin{equation*}
\underbrace{\sin ^{2} \angle B O D}_{l_{1}} \mu_{\mathrm{II}}^{2} \underbrace{-\frac{2 h^{2}}{r_{B}} \sin ^{2} \angle B O D}_{l_{2}} \mu_{\mathrm{II}} \underbrace{-\cos ^{2} \angle B O D v_{B}^{2} h^{2}+\frac{h^{4}}{r_{B}^{2}}}_{l 3}=0 \tag{4.40}
\end{equation*}
$$

Analytically solving for $\mu_{\text {II }}$ from Eq. (4.40), the charge product in Phase II is then obtained by Eq. (4.33).

Note that given $\mu_{\mathrm{I}}, t_{B}$ and $\angle B O D$, generally there are two solutions of $\mu_{\mathrm{II}}$ to Eq. (4.40). The solutions are

$$
\begin{align*}
& \mu_{\mathrm{II}}^{(1)}=\frac{h^{2}}{r_{B}}+\frac{1}{2 \sin ^{2} \angle B O D} \sqrt{l_{2}^{2}-4 l_{1} l_{3}}  \tag{4.41a}\\
& \mu_{\mathrm{II}}^{(2)}=\frac{h^{2}}{r_{B}}-\frac{1}{2 \sin ^{2} \angle B O D} \sqrt{l_{2}^{2}-4 l_{1} l_{3}} \tag{4.41b}
\end{align*}
$$

Substituting Eq. (4.41) into the RHS of Eq. (4.36), yields

$$
\begin{equation*}
\frac{h^{2}}{r}-\mu=\mp \frac{1}{2 \sin ^{2} \angle B O D} \sqrt{l_{2}^{2}-4 l_{1} l_{3}} \tag{4.42}
\end{equation*}
$$

This indicates that the two solutions result in two opposite signs in the RHS of Eq. (4.36). But for a particular value of $f$, either $-\angle B O D$ or $\pi-\angle B O D$, the LHS of Eq. (4.36) only has a specific sign. This means only one of the two solutions to Eq. (4.40) satisfies


Figure 4.4: Two cases of using $\mu_{\mathrm{II}}^{(1,2)}$ solutions, in both cases only one of the two solutions results in an actual symmetric trajectory.

Eq. (4.36). Physically speaking, only one of the two values in Eq. (4.41) will result in a symmetric trajectory.

The two plots in Figure 4.4 show the two scenarios using $\mu_{\mathrm{II}}^{(1,2)}$ given by Eq. (4.41). Figure $4.4(\mathrm{a})$ shows the case that we are intending to find a symmetric trajectory with the point D being the periapsis of Phase II, and Figure $4.4(\mathrm{~b})$ shows the case that the point D is designated as the apoapsis of Phase II. In the scenario in Figure $4.4(\mathrm{a})$, the angle $\angle B O D=71.9^{\circ}$, and $f_{B, \text { II }}$ is expected to be $-\angle B O D=-71.9^{\circ}$. With this value of $f_{B, \mathrm{II}}$, the LHS of Eq. (4.36) must be positive, correspondingly, only $\mu_{\mathrm{II}}^{(2)}$ satisfies Eq. (4.36). This is confirmed by Figure $4.4(\mathrm{a})$. Figure $4.4(\mathrm{~b})$ confirms the other fact that only $\mu_{\mathrm{II}}^{(1)}$ results in the symmetric trajectory with the point D being the apoapsis of Phase II.

By assuming the variables $\mu_{\mathrm{I}}$ and $t_{B}$ are given, the previous development outlines how to solve for the states at the points $B$ and $C$, and the charge product of Phase II $Q_{\mathrm{II}}$. However, in our present collision avoidance application $t_{B}$ is not explicity determined and will need to be solved using a numerical search routine. Note that three constraints have been used in deriving these formulas, Eqs. (4.5), (4.25), and (4.6). Next the safety
constraint in Eq. (4.10) needs to be utilized. The numerical search routine is intended to find an appropriate $t_{B}$ such that the closest distance $r_{\text {min }}=\gamma r_{s}$, where $\gamma \geq 1$.

The following theorem provides a rule to find the minimum distance $r_{\text {min }}$ in the whole trajectory.

Theorem 4 Consider the 3-Phase symmetric patched conic section trajectory as shown by Figure 4.2. If the point D is the periapsis of Phase II, then the minimum distance of the entire $\widehat{A E}$ trajectory is the periapsis radius of Phase II, i.e.:

$$
\begin{equation*}
r_{\min }=r_{p, \mathrm{II}} \tag{4.43}
\end{equation*}
$$

If the point D is the apoapsis of Phase II, then the minimum distance is the periapsis radius of Phase I, i.e.:

$$
\begin{equation*}
r_{\min }=r_{p, \mathrm{I}} \tag{4.44}
\end{equation*}
$$

Proof If D is the periapsis of Phase II, then $r_{p, \text { II }}$ is the minimum distance in Phase II. So it's true that $r_{p, \text { II }}<r_{B}$. Because $\angle B O D<90^{\circ}, f_{B, \text { II }} \in(-90,0)^{\circ}$, thus $\dot{r}_{B}<0$. Then the periapsis of Phase I does not lie along the arc $\widehat{A B}$. This indicates that throughout Phase I $\dot{r}<0$. Thus $r_{B}$ is the minimum distance in Phase I. Because $r_{p, \text { II }}<r_{B}, r_{p, \text { II }}$ is the minimum distance in the entire trajectory.

If D is the apoapsis of Phase II, then $r_{B}$ is the minimum distance in Phase II because $f_{B, \mathrm{II}} \in(90,180)^{\circ}$ and $\dot{r}_{B}>0$. Note that if $\dot{r}_{A}<0$, then the periapsis of Phase I must lie in the arc $\overparen{A B}$ because $\dot{r}$ crosses zero in Phase I. So $r_{p, \mathrm{I}}$ is the minimum distance in Phase I, this indicates that $r_{p, \mathrm{I}}<r_{B}$. Because $r_{B}$ is the minimum distance in Phase II, $r_{p, \mathrm{I}}$ is the minimum distance in the entire trajectory.

Theorem 4 states that if point D is the periapsis of Phase II, then the periapsis of Phase I must not lie on the arc $\overparen{A B}$. If point D is the apoapsis of Phase II, then the periapsis of Phase I must lie on the arc $\overparen{A B}$. Because the periapsis of Phase I lies in $\overparen{A B}, \dot{r}$ must cross zero in Phase I. Figure 4.5(b) illustrates this scenario in detail.

(a) Big scenario.

(b) Focused on Phase I-Phase II connecting points.

Figure 4.5: Illustration of the two cases with the point $D$ being the periapsis and apoapsis of Phase II.

Figure 4.6 shows the change of $r_{p, \text { II }}$ w.r.t. $t_{B}$ assuming that $\mu_{\mathrm{I}}$ is held fixed. The variable $t_{p, \mathrm{I}}$ is the time for SC 2 to fly from point A to the periapsis of Phase I. It can be seen that when $t_{B}<t_{p, \mathrm{I}}$ and $\mu_{\mathrm{II}}^{(2)}$ is used, $r_{p, \mathrm{II}}$ is monotonically increasing as $t_{B}$ increases; when $t_{B}>t_{p, \mathrm{I}}$ and $\mu_{\mathrm{II}}^{(1)}$ is used, $r_{p, \mathrm{II}}$ is monotonically decreasing as $t_{B}$ increases. Thus a symmetric collision avoidance trajectory with the point D being the periapsis of Phase II can be found by initializing $t_{B}^{(0)}<t_{p, \mathrm{I}}$ and updating $t_{B}$ using a common numerical methods such as Newton's method or the Secant method.

Alternatively initializing $t_{B}^{(0)}>t_{p, \mathrm{I}}$ and using $\mu_{\mathrm{II}}^{(1)}$ leads to a symmetric collision avoidance trajectory with the point D being the apoapsis of Phase II. Note that this solution yields $r_{p, \text { II }}=\gamma r_{s}$ which is a conservative maneuver because the point D is the apoapsis of Phase II. Another way to achieve the solution with the point D being the apoapsis is to set $r_{p, \mathrm{I}}=\gamma r_{s}$, and solve for corresponding $\mu_{\mathrm{I}}$ from Eq. (4.21). Any symmetric trajectory with the point D being the apoapsis of Phase II will satisfy collision avoidance requirement. Thus there are infinite choices of $t_{B}$ which lead to symmetric maneuvers and varying apoapses.

Before performing a numerical search for $t_{B}$ for a given $\mu_{\mathrm{I}}$ it must be decided


Figure 4.6: Given $\mu_{\mathrm{I}}$, the resulting $r_{p, \text { II }}$ w.r.t. $t_{B}$.
apriori if a periapses or apoapses $D$ point solution is sought. During the numerical iterations the current estimates of $t_{B}$ must be constrained to remain either larger or smaller than the periapses time $t_{p, \mathrm{I}}$ of Phase I. If $t_{B}$ crosses $t_{p, \mathrm{I}}$ without switching the $\mu_{\text {II }}$ solution, the algorithm will lead to an asymmetric trajectory with $f_{B, \text { II }}$ lying in a wrong quadrant, as shown by the dashed lines in Figure 4.4.

Note that the path with the point D being the apoapsis of Phase II is a longer path, both in length and in time. Practically speaking, there is a bigger chance for the longer path to be influenced by disturbances. Though in developing the algorithm the Debye length effect is not taken into consideration, this effect does exist in the space environment. Since the longer path will be influenced more due to disturbance, the shorter path with the point D being the periapsis should be prefered.

Finally all the required sub-steps have been presented to outline the overall collision avoidance algorithm. The basic logic is to search for a proper $t_{B}^{*}$ such that the collision avoidance criteria

$$
\begin{equation*}
r_{p, \mathrm{II}}=\gamma r_{s} \tag{4.45}
\end{equation*}
$$

is satisfied, with the point D being the periapsis of Phase II. If for some reason $t_{B}^{*}$ is not achievable, for example if $t_{B}^{*}$ is so short that the spacecraft have missed it already
at time $t_{B}^{*}$, the algorithm switches to find a circular transitional trajectory.
In this chapter, Newton's method is used in the numerical searching for $t_{B}^{*}$ such that the following target function becomes zero:

$$
\begin{equation*}
g\left(t_{B}\right)=r_{p, \mathrm{II}}\left(t_{B}\right)-\gamma r_{s} \tag{4.46}
\end{equation*}
$$

The iteration algorithm to determine a symmetric collision avoidance with D being the periapses of Phase II propagates according to the following steps:

Step 1 Initialization: From the measurements $\boldsymbol{r}_{A}$ and $\dot{\boldsymbol{r}}_{A}$, calculate $e_{I}, a_{I}$ through Eq. (4.14), and calculate the angle $\angle A O D$ through Eq. (4.15). Prescribe a proper $\mu_{\mathrm{I}}$, which means $\left|\mu_{\mathrm{I}}\right|$ must be greater than $\left|\mu_{\mathrm{I}, \mathrm{c}}\right|$ to ensure $r_{p, \mathrm{I}}>\gamma r_{s}$. It must also make sure $Q_{\mathrm{I}}$ is implementable, which means $Q_{\mathrm{I}}<Q_{\max }$. Calculate $t_{p, \mathrm{I}}$. Initialize $t_{B}$ :

$$
\begin{equation*}
t_{B}^{(0)}=\alpha t_{p, \mathrm{I}} \tag{4.47}
\end{equation*}
$$

where $0<\alpha<1$.
Step 2 Solve for the point $B$ 's states $r_{B}$ and $v_{B}$ through Eqs. (4.26)-(4.29).
Step 3 Solve for $\mu_{\mathrm{II}}^{(2)}$ by Eq. (4.41), using the minus sign. Calculate $r_{p, \mathrm{II}}$ through

$$
\begin{equation*}
r_{p, \mathrm{II}}=a_{\mathrm{II}}\left(1-e_{\mathrm{II}}\right) \tag{4.48}
\end{equation*}
$$

and $a_{\mathrm{II}}$ is solved by the energy equation, $e_{\mathrm{II}}$ is calculated through Eq. (4.34) evaluating at point $B$ in Phase II.

Step 4 Calculate $g\left(t_{B}\right)$ by Eq. (4.46). Judge whether $\left|g\left(t_{B}\right)\right|<$ Tol. If yes, STOP. Otherwise, go to Step 5.

Step 5 Calculate $g^{\prime}=\frac{\partial g}{\partial t_{B}}$ using the finite difference method.
Step 6 Update $t_{B}^{(i+1)}=t_{B}^{(i)}-\frac{g}{g^{\prime}}, i=i+1$. Go to Step 2.


Figure 4.7: Geometry of the 2-spacecraft system.

After choosing a proper value of $Q_{\mathrm{I}}$, this routine calculates a symmetric collision avoidance trajectory composed of three patched conic-sections.

### 4.5 Collision Avoidance Criteria with Charge Saturation

The previous section develops a numerical routine to find a symmetric patched conic section trajectory to avoid the collision and meanwhile preserve the relative velocity magnitude and direction of the two-spacecraft system. In deriving these routines, it is assumed that the charge product of the two spacecraft is unlimited. If the charge product limitation is taken into consideration, the system's ability to avoid a potential collision is then limited. Under certain conditions, for example the two spacecraft are approaching each other too quickly, the collision would be unpreventable. This section is intended to determine criteria to predict whether a potential collision can be prevented using the presented collision avoidance routines.

Figure 4.7 illustrates the geometry of the two spacecraft system when the collision avoidance strategy is triggered at time $t_{A}$. The vectors $\boldsymbol{r}_{A}, \boldsymbol{v}_{A}$ and $\boldsymbol{h}$ can be expressed
in the $\left\{\hat{\boldsymbol{\imath}}_{v}, \hat{\boldsymbol{\imath}}_{h}, \hat{\boldsymbol{\imath}}_{D}\right\}^{1}$ frame as

$$
\begin{align*}
\boldsymbol{r}_{A} & =-x_{A} \hat{\boldsymbol{\imath}}_{v}+d \hat{\boldsymbol{\imath}}_{D}  \tag{4.49a}\\
\boldsymbol{v}_{A} & =v_{0} \hat{\boldsymbol{\imath}}_{v}  \tag{4.49b}\\
\boldsymbol{h} & =\boldsymbol{r}_{A} \times \boldsymbol{v}_{A}=d v_{0} \hat{\boldsymbol{\imath}}_{h} \tag{4.49c}
\end{align*}
$$

This section is investigating the critical state with $\gamma=1$. Substituting Eq. (4.49) into Eq. (4.21), and using the fact $\left\|\boldsymbol{r}_{A}\right\|=r_{o}$, yields

$$
\begin{equation*}
\mu_{\mathrm{I}, \mathrm{c}}=\frac{r_{o} v_{0}^{2} d^{2}-r_{s}^{2} r_{o} v_{0}^{2}}{2 r_{s}\left(r_{o}-r_{s}\right)} \tag{4.50}
\end{equation*}
$$

Eq. (4.50) privides the value of $\mu_{\mathrm{I}}$ that results in $r_{p, \mathrm{I}}=r_{s}$. Thus the circular transitional orbit solution gives $\mu_{\mathrm{I}}$ in the critical state.

Theorem 5 Consider a repulsive hyperbola motion governed by Eq. (4.4), with $\mu<0$ being constant. Given initial position and velocity $\left[\boldsymbol{r}_{0}, \dot{\boldsymbol{r}}_{0}\right]$, the radius of the periapsis $r_{p}$ increases as $|\mu|$ increases.

Proof To mathematically prove this theorem, it's required to express $r_{p}$ in terms of $\mu$ and initial conditions. For a repulsive hyperbola, the periapsis radius is given as [12]

$$
\begin{equation*}
r_{p}=a(1+e) \tag{4.51}
\end{equation*}
$$

Here $a$ and $e$ are actually determined by the initial conditions and $\mu$. Substituting $e=\sqrt{1-h^{2} / \mu a}$ and Eq. (4.14b) into Eq. (4.51) and using $|\mu|=-\mu$ instead of $\mu$, yield

$$
\begin{equation*}
r_{p}=\frac{1}{2|\mu| / r_{0}+v_{0}^{2}}\left(|\mu|+\sqrt{|\mu|^{2}+h^{2}\left(2|\mu| / r_{0}+v_{0}^{2}\right)}\right) \tag{4.52}
\end{equation*}
$$

where $r_{0}=\left\|\boldsymbol{r}_{0}\right\|, v_{0}=\left\|\dot{\boldsymbol{r}}_{0}\right\|, h=\left\|\boldsymbol{r}_{0} \times \dot{\boldsymbol{r}}_{0}\right\|$, which are all determined by the initial conditions.

[^1]It's still not obvious to see the trend of $r_{p}$ as $|\mu|$ increases. Taking a partial derivative of $r_{p}$ with respect to $|\mu|$, yields

$$
\begin{equation*}
\frac{\partial r_{p}}{\partial|\mu|}=\frac{1+\left(|\mu|+h^{2} / r_{0}\right) / \beta}{2|\mu| / r_{0}+v_{0}^{2}}-\frac{|\mu|+\beta}{r_{0}\left(2|\mu| / r_{0}+v_{0}^{2}\right)^{2}} \tag{4.53}
\end{equation*}
$$

where $\beta=\sqrt{|\mu|^{2}+h^{2}\left(2|\mu| / r_{0}+v_{0}^{2}\right)}$. The trend of $r_{p}$ as $\mu$ increases is determined by the sign of $\frac{\partial r_{p}}{\partial|\mu|}$. Eq. (4.53) can be changed to be:

$$
\begin{equation*}
\frac{\partial r_{p}}{\partial|\mu|}=\frac{1}{\left(2|\mu| / r_{0}+v_{0}^{2}\right)^{2}}\left\{\frac{|\mu|}{r_{0}}+v_{0}^{2}+\left(\frac{|\mu|^{2}}{r_{0}}+|\mu| v_{0}^{2}\right) / \beta\right\} \tag{4.54}
\end{equation*}
$$

Eq. (4.54) gives a simplified expression of $\frac{\partial r_{p}}{\partial|\mu|}$ with every individual term being positive. Thus the partial derivative $\frac{\partial r_{p}}{\partial|\mu|}$ is alway positive. This proves that $r_{p}$ increases as $|\mu|$ increases.

Applying Theorem 5 in the 3-Phase symmetric patched conic section scenario, yields the following lemma.

Lemma 1 For the 3-Phase patched conic section scenario as shown in Figure 4.2, the circular transitional trajectory solution provides the minimum value of $Q_{\mathrm{I}}$ that satisfies the collision avoidance constraint $r_{\min } \geq r_{s}$.

Proof For the critical case where $\gamma=1$, the circular transitional trajectory has the following properties:

$$
\begin{equation*}
r_{p, \mathrm{I}}=r_{s}, \quad r_{\mathrm{II}}=r_{s} \tag{4.55}
\end{equation*}
$$

where $r_{\text {II }}$ is the radius of Phase II which is constant.
By Theorem 5, $\mu_{\mathrm{I}, \mathrm{c}}$ in Eq. (4.50) provides the minimum value of $\left|\mu_{\mathrm{I}}\right|$ that satisfies $r_{p, \mathrm{I}} \geq r_{s}$. From Eq. (4.3), the charge product $Q_{\mathrm{I}}$ is proportional to $\left|\mu_{\mathrm{I}}\right|$, thus the circular transitional trajectory provides the minimum value of $Q_{\mathrm{I}}$ such that $r_{p, \mathrm{I}} \geq r_{s}$. For Phase II, the radius is equal to $r_{s}$, which satisfies the collision avoidance requirement. So the circular transitional trajectory solution provides the minimum $Q_{\mathrm{I}}$ to avoid the collision.

Theorem 6 For the two effective gravitational coefficients given by Eq. (4.21) and Eq. (4.22), $\mu_{\mathrm{I}, c}>\left|\mu_{\mathrm{II}, c}\right|$ if and only if $d<d^{*}=r_{s} \sqrt{\frac{r_{o}}{3 r_{o}-2 r_{s}}}$.

Proof First let us investigate $\mu_{\mathrm{II}, \mathrm{c}}-\left|\mu_{\mathrm{I}, \mathrm{c}}\right|$ :

$$
\begin{align*}
\mu_{\mathrm{II}, \mathrm{c}}-\left|\mu_{\mathrm{I}, \mathrm{c}}\right| & =\mu_{\mathrm{II}, \mathrm{c}}+\mu_{\mathrm{I}, \mathrm{c}} \\
& =\frac{h^{2}}{r_{s}}+\frac{r_{o} v_{0}^{2} d^{2}-r_{s}^{2} r_{o} v_{0}^{2}}{2 r_{s}\left(r_{o}-r_{s}\right)} \\
& =\frac{v_{0}^{2}}{2 r_{s}\left(r_{o}-r_{s}\right)}\left(\left(3 r_{o}-2 r_{s}\right) d^{2}-r_{o} r_{s}^{2}\right) \tag{4.56}
\end{align*}
$$

When $\left|\mu_{\mathrm{I}, \mathrm{c}}\right|>\mu_{\mathrm{II}, \mathrm{c}}, \mu_{\mathrm{II}, \mathrm{c}}-\left|\mu_{\mathrm{I}, \mathrm{c}}\right|<0$, applying this to the formula in Eq. (4.56), yields

$$
\begin{equation*}
\frac{v_{0}^{2}}{2 r_{s}\left(r_{o}-r_{s}\right)}\left(\left(3 r_{o}-2 r_{s}\right) d^{2}-r_{o} r_{s}^{2}\right)<0 \quad \Leftrightarrow \quad d<\sqrt{\frac{r_{o} r_{s}^{2}}{3 r_{o}-2 r_{s}}}=d^{*} \tag{4.57}
\end{equation*}
$$

Theorem 5 and Lemma 1 show $Q_{\mathrm{I}}$ is lower bounded by the circular Phase II solution:

$$
\begin{equation*}
Q_{\mathrm{I}} \geq Q_{\mathrm{I}, \mathrm{c}}=-\frac{\mu_{\mathrm{I}, \mathrm{c}} m_{1} m_{2}}{k_{\mathrm{c}}\left(m_{1}+m_{2}\right)}=-\frac{\left(r_{o} v_{0}^{2} d^{2}-r_{s}^{2} r_{o} v_{0}^{2}\right) m_{1} m_{2}}{2 k_{\mathrm{c}} r_{s}\left(r_{o}-r_{s}\right)\left(m_{1}+m_{2}\right)} \tag{4.58}
\end{equation*}
$$



Figure 4.8: Charge product values under different $d$.

When implementing these charge collision avoidance solutions, the averaged charge is less of a concern because the spacecraft charge can be servoed with very little electrical power and using essentially no fuel. [1] Instead, the required absolute levels should be made as small as possible. This results in smaller required spacecraft potentials and less issues with electrostatic discharges. Assuming that $d$ satisfies the condition in Theorem 6, then note that the circular transfer orbit provides the minimum $Q_{\mathrm{I}}$ collision avoidance solution. To illustrate this, consider the following numerical simulation results with the initial conditions:

$$
\begin{equation*}
\boldsymbol{R}_{1}\left(t_{0}\right)=[0,0,0]^{T} \mathrm{~m}, \dot{\boldsymbol{R}}_{1}\left(t_{0}\right)=[0,0,0]^{T} \mathrm{~m} / \mathrm{s}, \boldsymbol{R}_{2}\left(t_{0}\right)=[20, d, 0]^{T} \mathrm{~m}, \dot{\boldsymbol{R}}_{2}\left(t_{0}\right)=[-0.03,0,0]^{T} \mathrm{~m} / \mathrm{s} \tag{4.59}
\end{equation*}
$$

and with $r_{o}=15 \mathrm{~m}, r_{s}=5 \mathrm{~m}$. Figure 4.8 shows the charge product values under different values of the offset distance $d$. For the circular transitional trajectory case Theorem 6 states that $Q_{\mathrm{I}}>Q_{\mathrm{II}}$ when $d<3.2733 \mathrm{~m}$, and this is reflected in Figure 4.8(a).

For general symmetric trajectory cases, given a value of $d$, there remains one degree of freedom to determine the collision avoidance trajectory. In the numerical algorithm presented before we can choose a value of $Q_{\mathrm{I}}$ and then calculate all the remaining variables. Figure $4.8(\mathrm{~b})$ shows the value of $\left|Q_{\mathrm{II}}\right|$ corresponding to $Q_{\mathrm{I}}$ under different $d$, with all other variables the same as in Figure 4.8(a). The shaded area is the region where $\left|Q_{\mathrm{II}}\right|>Q_{\mathrm{I}}$.

Figure 4.8(b) illustrates that the solution with $\left|Q_{\mathrm{II}}\right|<Q_{\mathrm{I}}$ always exists, while the solution with $\left|Q_{\mathrm{II}}\right|>Q_{\mathrm{I}}$ exists only when $d>d^{*}$. This agrees with intuition because $Q_{\mathrm{I}}$ can be infinitely large to achieve the symmetric collision avoidance trajectory, but it must be greater than a certain value to ensure a collision avoidance with $r>r_{s}$. When $d<d^{*}$, the minimum acceptable value of $Q_{\mathrm{I}}$ is still greater than corresponding $\left|Q_{\mathrm{II}}\right|$ as predicted by Theorem 6, thus the solution with $\left|Q_{\mathrm{II}}\right|>Q_{\mathrm{I}}$ does not exist in this situation. Another important thing is that if $d<d^{*}$, the solution with $Q_{\mathrm{I}}=\left|Q_{\mathrm{II}}\right|$ is the
$L_{\infty}$ optimal charge solution; when $d>d^{*}$, the circular transfer orbit is the $L_{\infty}$ charge optimal solution. This helps to choose a proper value of $Q_{\mathrm{I}}$ such that the maximum charge level during the whole process is minimized.

Note that the criteria in Eq. (4.58) has exactly the same form as Eq. (42) in Chapter 3. That equation is the requirement for the charge product such that the collision can be avoided. It has the same form as Eq. (4.58) because Chapter 3 assumes that the two spacecraft are fully charged to get the criteria in Eq. (42). This assumption matches with the situation in Phase I, where the two spacecraft have a constant charge product and are repeling each other. The physical meanings of the criteria in Eq. (42) in Chapter 3 can also be utilized here. For a given formation flying mission in which the maximum magnitude of the possible separation distance rate has been determined, Eq. (4.58) provides a guide to design the spacecraft charge devices such that $Q_{\mathrm{I}, \mathrm{c}}$ is achieveable, thus the collision can be avoided with a symmetric trajectory.

If the maximum charge product has been specified, then Eq. (4.60) below tells us the maximum allowable relative velocity that guarantees the collision to be avoidable.

$$
\begin{equation*}
v_{0} \leq \sqrt{\frac{2 Q_{\mathrm{I}, \max } k_{\mathrm{c}}\left(m_{1}+m_{2}\right)}{m_{1} m_{2}} \frac{r_{s}\left(r_{o}-r_{s}\right)}{r_{o}\left(d^{2}-r_{s}^{2}\right)}} \tag{4.60}
\end{equation*}
$$

Note that the inequality in Eq. (4.60) is obtained by solving for $v_{0}$ from the inequality in Eq. (4.58).

### 4.6 Numerical Simulations

A numerical iteration routine using Newton's method to solve for a symmetric patched conic section trajectory has been setup. The basic logic of the routine is to first search an appropriate time value $t_{B}$ such that the target function $g\left(t_{B}\right)$ defined in Eq. (4.46) converges to zero, and the point D is the periapsis of Phase II.

The following numerical simulation cases show the effectiveness of the routine in different situations. All the cases share a common set of the parameters of the two
spacecraft system:

$$
\begin{equation*}
m_{1}=m_{2}=50 \mathrm{~kg}, \quad r_{o}=15 \mathrm{~m}, \quad r_{s}=7 \mathrm{~m}, \quad \gamma=1 . \tag{4.61}
\end{equation*}
$$

The initial inertial state vectors are also the same across all numerical studies unless specified:

$$
\left\{\begin{array} { l } 
{ \boldsymbol { R } _ { 1 } ( t _ { 0 } ) = [ 0 , 0 , 0 ] ^ { T } \mathrm { m } }  \tag{4.62}\\
{ \boldsymbol { R } _ { 2 } ( t _ { 0 } ) = [ - 1 6 , 3 , 0 ] ^ { T } \mathrm { m } }
\end{array} \quad \left\{\begin{array}{l}
\dot{\boldsymbol{R}}_{1}\left(t_{0}\right)=[0,0,0]^{T} \mathrm{~m} / \mathrm{s} \\
\dot{\boldsymbol{R}}_{2}\left(t_{0}\right)=[0.02,0,0]^{T} \mathrm{~m} / \mathrm{s}
\end{array}\right.\right.
$$

### 4.6.1 Ideal Conditions Examples

The phrase "ideal conditions" means the two spacecraft are flying in free space in a vacuum (not plasma environment) with $\lambda_{d}=\infty$. Setting the variable $\mu_{\mathrm{I}}=-0.01 \mathrm{~m}^{3} / \mathrm{s}^{2}$, the corresponding charge product is $Q_{\mathrm{I}}=27.81 \mu \mathrm{C}^{2}$. Under these conditions, Figure 4.9 shows two simulation results. The first trajectory has the point D as the periapsis of Phase II. The second one is the case that the point D is the apoapsis of Phase II. This can be achieved by initializing $t_{B}$ to be larger than $t_{p, \mathrm{I}}$, and using $\mu_{\mathrm{II}}^{(1)}$ instead of $\mu_{\mathrm{II}}^{(2)}$ in the routine. Table 4.1 shows some detailed results of the simulations.


Figure 4.9: Idea simulation.

In both of the two simulations, the collision avoidance requirement $r_{\min } \geq r_{s}$ is satisfied. and the final relative speed direction is held the same as the initial direction.

Table 4.1: Results of the ideal simulations.

|  | $t_{B}[\mathrm{~s}]$ | $r_{D}[\mathrm{~m}]$ | $Q_{\mathrm{II}}\left[\mu \mathrm{C}^{2}\right]$ |
| :--- | :---: | :---: | :---: |
| $\operatorname{sim} 1$ | 291.42 | 7.00 | -6.480 |
| $\operatorname{sim} 2$ | 391.61 | 15.64 | -1.036 |

The first simulation has the shorter path, though the magnitude of $Q_{\text {II }}$ is bigger. The apoapsis case (means the case with the poind D being the apoapsis of Phase II) is very conservative. Noticing that a small difference in $t_{B}$ results in a huge difference in the total maneuver time, the longer transition time span makes the apoapsis case much more vulnerable to disturbances.

### 4.6.2 Charge Expense Analysis

In this simulation case, the charge expense under different value of $\mu_{\mathrm{I}}$ is analyzed. According to different concerns about charge expense, the two charge cost functions $J_{1}$ and $J_{2}$ are defined below:

$$
\begin{equation*}
J_{1}=\max \left(Q_{\mathrm{I}},\left|Q_{\mathrm{II}}\right|\right), \quad J_{2}=\frac{2 t_{B} Q_{\mathrm{I}}+t_{\mathrm{II}}\left|Q_{\mathrm{II}}\right|}{2 t_{B}+t_{\mathrm{II}}} \tag{4.63}
\end{equation*}
$$

Here $J_{1}$ is the maximum magnitude of the charge products. It is important when the maximum vehicle voltage level is of concern. $J_{2}$ is the time averaged charge product which provides insight into the nominal charge and voltages levels. Numerical sweeps on $\left|\mu_{\mathrm{I}}\right|$ are performed using the same parameters as in Eq. (4.61), but with the different initial conditions:

$$
\left\{\begin{array} { l } 
{ \boldsymbol { R } _ { 1 } ( t _ { 0 } ) = [ 0 , 0 , 0 ] ^ { T } \mathrm { m } }  \tag{4.64}\\
{ \boldsymbol { R } _ { 2 } ( t _ { 0 } ) = [ - 1 6 , 6 , 0 ] ^ { T } \mathrm { m } }
\end{array} \quad \left\{\begin{array}{l}
\dot{\boldsymbol{R}}_{1}\left(t_{0}\right)=[0,0,0]^{T} \mathrm{~m} / \mathrm{s} \\
\dot{\boldsymbol{R}}_{2}\left(t_{0}\right)=[0.03,0,0]^{T} \mathrm{~m} / \mathrm{s}
\end{array}\right.\right.
$$

Note that with the provided parameters and initial conditions, the condition in Theorem 6 is not satisfied, which implies the solution with $\left|Q_{\mathrm{II}}\right|>Q_{\mathrm{I}}$ exists. Figure 4.10 shows the values of $J_{1}$ and $J_{2}$ for each value of $\left|\mu_{\mathrm{I}}\right|$. Figure 4.10(a) shows that the minimum value of $J_{1}$ is achieved at the marked point where $Q_{\mathrm{I}}=\left|Q_{\mathrm{II}}\right|$. As $\left|\mu_{\mathrm{I}}\right|$ increases,


Figure 4.10: Charge expense history while sweeping $\mu_{\mathrm{I}}$.
before it reaches the point where $\left|Q_{\mathrm{II}}\right|=Q_{\mathrm{I}},\left|Q_{\mathrm{II}}\right|$ dominates and $J_{1}=\left|Q_{\mathrm{II}}\right|$. After the marked point, $J_{1}$ is linearly increasing because now $J_{1}=Q_{\mathrm{I}}$ and $Q_{\mathrm{I}}$ is proportional to $\left|\mu_{\mathrm{I}}\right|$. Figure $4.10(\mathrm{~b})$ shows that the minimum $J_{2}$ happens at the point where $Q_{\mathrm{I}}$ is minimum. This is because when $Q_{\mathrm{I}}=Q_{\mathrm{I}, c}, 2 t_{B}$ is about two times greater than $t_{\mathrm{II}}$, and as $\left|\mu_{\mathrm{I}}\right|$ increases, $t_{B}$ is increasing and $t_{\mathrm{II}}$ is decreasing. Thus the influence of $t_{B}$ dominates $J_{2}$.

The two plots in Figure 4.10 together show an example that according to difference charge expense concerns, the "optimal" solutions can be different.

### 4.6.3 Simulation With Debye Length Effect

The algorithm developed in this chapter is an open loop programming algorithm, assuming that the spacecraft are flying in free space which implies the orbital motion and Debye length effect haven't been taken into account. Figure 4.11 shows the difference when the algorithm is directly applied in the simulation that the environment Debye length $\lambda_{d}=50 \mathrm{~m}$. This value represents the Debye length in deep space at 1 AU distance from the sun.

The final velocity direction of the disturbed trajectory has an offset of $3.98^{\circ}$ from


Figure 4.11: Relative trajectories of the two spacecraft under the condition $\lambda_{d}=50 \mathrm{~m}$.
the ideal trajectory. The minimum distance of the disturbed trajectory is 0.254 m or about $3.6 \%$ less than that of the ideal case $\gamma r_{s}$, due to the partial shielding of Coulomb force. The Debye length always decreases the effectiveness of the Coulomb repulsion. This effect could be compensated for with a $\gamma>1$ safety factor. Future work will investigate how to feedback stabilize such open-loop trajectories. A challenge here is the under-actuated nature of the Coulomb thrusting. Further, the momentum conservation makes it impossible to reverse the motion to compensate for an overshoot. Any feedback control development could try to bias the tracking errors to slightly undershoot the desired trajectory.

### 4.7 Conclusion

This chapter develops an open-loop trajectory programming algorithm to find a symmetric trajectory composed of three patched conic-sections to avoidance a potential collision. Compareing to the feedback charge control strategy developed in Chapter 3, this approach is able to match both the direction and the magnitude of the relative motion speed with the initial relative approach velocity vector. At first a circular transitional trajectory is obtained analytically. This solution provides the minimum charge product magnitude that ensures a collision avoidance. Assuming a value of the charge product in Phase I, a numerical routine is developed to find a symmetric three-conic-section trajectory by using the collision avoidance requirement and the symmetric constraints. The dual-solution problem for the effective gravitational parameter is in-
vestigated to analytically determine which results in a symmetric trajectory. The rule in choosing the correct solution is that the switching time between Phase I and Phase II can not cross the periapsis time of Phase I during the numerical iteration. The criteria for a collision avoidance by using the symmetric trajectory algorithm with charge saturation are found by investigating the geometries of the two-spacecraft system. Numerical simulations show that under different definitions of the cost function, the optimal value for the charge product in Phase I varies. This implies that a routine to search for optimal charge product in Phase I can be applied catering to a specific definition of the cost function to initialize the algorithm presented in the chapter, which is used to find the general symmetric collision avoidance trajectory.

The idea of generating the patched conic trajectory can be applied to more general missions, such as two spacecraft fly-by maneuver. The scenario of fly-by maneuver is that, two spacecraft have their initial position and velocity vectors resulting a certain relative motion. Due to some reasons, now we want to change the relative motion between the two spacecraft to some desired motion. The desired relative motion is given by a relative speed vector. During the maneuver, the two spacecraft must not collide. A possible approach is very similar to the symmetric trajectory programming method used in collision avoidance. There are two obvious differences: the symmetric axis should be tilted corresponding to the direction of the desired relative speed; Phase-I and Phase-III may not have the same charge level according to the magnitude of the desired relative speed. This research is very interesting and doable. It can be even applied to capture the energy from a fly-by spacecraft which is very similar to the "Gravity Assistant" concept used in interplanetary space missions.

## CHAPTER 5

## NONLINEAR CONTROL OF A TWO-CRAFT COULOMB VIRTUAL STRUCTURE

Before investigating the three-craft Coulomb virtual structure, it's necessary to have in-depth knowledge about the simpler control case of a Coulomb virtual structure with only two spacecraft. This chapter studies the Coulomb virtual structure control problem in the scenario that a two-spacecraft formation operates in a Geostationary Earth Orbit (GEO). Of interest is if the 2-craft shape controlled requires full-state feedback, or if this shape control is possible with partial state feedback of the separation distance information only. The work in this chapter has been presented in Reference [29] and has been submitted to IEEE Transaction on Aerospace and Electronic System for publication.


Figure 5.1: Scenario of the 2 spacecraft system.

### 5.1 Equations Of Motion

The actively controlled Coulomb force between the spacecraft is the only force utilized to control the separation distance. No hybrid thrusting (blend of Coulomb and conventional inertial thrusting) is considered. Note that the Coulomb forces cannot directly change the inertial angular momentum of the system because they are system-internal forces. Instead, the objective of the control is to maintain the separation distance to be a certain desired value such that the shape of the two-body formation is held nominally constant.

Assuming the spacecraft potential is small compared to the local plasma kinetic energy, the Coulomb force between the two spacecraft acting on spacecraft-1 (SC-1) is approximated as:

$$
\begin{equation*}
\boldsymbol{F}_{c}=-k_{c} \frac{Q}{L^{2}}\left(1+\frac{L}{\lambda_{\mathrm{d}}}\right) e^{-\frac{L}{\lambda_{\mathrm{d}}}} \hat{\boldsymbol{e}}_{r} \tag{5.1}
\end{equation*}
$$

where $k_{c}=8.99 \times 10^{9} \mathrm{Nm}^{2} \mathrm{C}^{-2}$ is the Coulomb constant, $Q$ is the charge product of the two spacecraft, $L=\|\boldsymbol{r}\|$ is the separation distance between the two spacecraft, $\hat{e}_{r}=\boldsymbol{r} / L$ is the unit vector pointing from SC-1 to SC-2, $\lambda_{\mathrm{d}}$ is the Debye length characterizing the plasma shielding effect.

The inertial equations of motion (EOM) are given by

$$
\begin{align*}
& m_{1} \ddot{\boldsymbol{R}}_{1}=-\frac{G M m_{1}}{R_{1}^{2}} \hat{\boldsymbol{o}}_{r 1}-k_{c} \frac{Q}{L^{2}}\left(1+\frac{L}{\lambda_{\mathrm{d}}}\right) e^{-\frac{L}{\lambda_{\mathrm{d}}} \hat{\boldsymbol{e}}_{r}}  \tag{5.2a}\\
& m_{2} \ddot{\boldsymbol{R}}_{2}=-\frac{G M m_{2}}{R_{2}^{2}} \hat{\boldsymbol{o}}_{r 2}+k_{c} \frac{Q}{L^{2}}\left(1+\frac{L}{\lambda_{\mathrm{d}}}\right) e^{-\frac{L}{\lambda_{\mathrm{d}}} \hat{\boldsymbol{e}}_{r}} \tag{5.2b}
\end{align*}
$$

where $G=6.67428 \times 10^{-11} \mathrm{~m}^{3} \mathrm{~kg}^{-1} \mathrm{~s}^{-2}$ is the gravitational constant, $M=5.9736 \times 10^{24}$ kg is the Earth's mass. The states $\boldsymbol{R}_{i}, m_{i}$ and $q_{i}$ are the inertial position vector, the mass and the charge of the $i^{\text {th }}$ spacecraft respectively, while $\hat{\boldsymbol{o}}_{r i}=\boldsymbol{R}_{i} / L_{i}$ is the unit vector of the inertial position vector of the $i^{\text {th }}$ spacecraft.

In order to develop a control algorithm to stabilize the separation distance (i.e. the virtual structure shape) of the two spacecraft, we derive the separation distance
equation of motion.Using Eq. (5.2), the relative EOM is:

$$
\begin{equation*}
\ddot{\boldsymbol{r}}=\ddot{\boldsymbol{R}}_{2}-\ddot{\boldsymbol{R}}_{1}=\frac{G M}{R_{1}^{2}} \hat{\boldsymbol{o}}_{r 1}-\frac{G M}{R_{2}^{2}} \hat{\boldsymbol{o}}_{r 2}+k_{c} \frac{Q}{L^{2}}\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right)\left(1+\frac{L}{\lambda_{d}}\right) e^{-\frac{L}{\lambda_{d}}} \hat{\boldsymbol{e}}_{r} \tag{5.3}
\end{equation*}
$$

Differentiating the identity $L=\sqrt{\boldsymbol{r} \cdot \boldsymbol{r}}$ twice yields the separation distance acceleration relationship:

$$
\begin{equation*}
\ddot{L}=\ddot{\boldsymbol{r}} \cdot \hat{\boldsymbol{e}}_{r}+\frac{1}{L}\|\dot{\boldsymbol{r}}\|^{2}\left(1-\cos ^{2} \angle(\boldsymbol{r}, \dot{\boldsymbol{r}})\right) \tag{5.4}
\end{equation*}
$$

Substituting Eq. (5.3) into Eq. (5.4) yields the desired separation distance EOM:

$$
\begin{align*}
\ddot{L}=k_{c} \frac{Q}{L^{2}} & \left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right)\left(1+\frac{L}{\lambda_{d}}\right) e^{-\frac{L}{\lambda_{d}}}+\underbrace{G M\left(\frac{1}{R_{1}^{2}} \hat{\boldsymbol{o}}_{r 1}-\frac{1}{R_{2}^{2}} \hat{\boldsymbol{o}}_{r 2}\right) \cdot \hat{\boldsymbol{e}}_{r}}_{f_{1}} \\
& +\underbrace{\frac{1}{L}\|\dot{\boldsymbol{r}}\|^{2}\left(1-\cos ^{2} \angle(\boldsymbol{r}, \dot{\boldsymbol{r}})\right)}_{f_{2}} \tag{5.5}
\end{align*}
$$

Note that the term $f_{1}$ is a function of the inertial position vectors of the formation, while $f_{2}$ is solely a function of the relative position vectors of the formation.

### 5.2 Two-Craft Shape Control Algorithm

The goal of this chapter is to develop a static shape control of a spinning charged two-spacecraft formation. The control objective is thus only the shape of the formation, not the orientation of the formation. This section develops a Lyapunov-based nonlinear controller to make the separation distance of the two spacecraft stabilized at the desired distance. Let us define a shape error as

$$
\begin{equation*}
\Delta x=L-L^{*} \tag{5.6}
\end{equation*}
$$

where $L^{*}$ is the desired constant distance. The objective of the control is to make $\Delta x \rightarrow 0$. Because the desired distance $L^{*}$ is constant, the relative trajectory of the two body system is circular. For a two body Coulomb formation with separation distance within 100 m , the satellites' major accelerations is due to the Coulomb forces. Thus, after the distance error converges, the control charge would be a constant value that
maintains the shape of the spinning structure. This chapter defines the control charge product as a summation of a feed-forward and a feedback component:

$$
\begin{equation*}
Q=Q_{n}+\delta Q \tag{5.7}
\end{equation*}
$$

Here $Q_{n}$ is the feed-forward control component that maintains the shape of the final spinning structure, $\delta Q$ is the feedback part that stabilizes the distance error.

### 5.2.1 Spinning Two-Craft Feed-Forward Control

The feed-forward control is obtained by finding the equilibrium solution of the control charge product under the assumption that the two spacecraft are flying in deep space. This way the influence of the planetary gravity is treated as a disturbance that is taken care of by the feedback control part. Neglecting the planetary gravity influences, the EOM in Eq. (5.5) becomes

$$
\begin{equation*}
\ddot{L}^{*}=k_{c} \frac{Q}{L^{* 2}}\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right)\left(1+\frac{L^{*}}{\lambda_{d}}\right) e^{-\frac{L^{*}}{\lambda_{d}}}+f_{2}^{*} \tag{5.8}
\end{equation*}
$$

where $f_{2}^{*}$ is the ideal value of $f_{2}$ when the distance error converges to zero. Forcing $\ddot{L}=0$ yields the feed-forward control charge product:

$$
\begin{equation*}
Q_{n}=-\frac{L^{* 2} \lambda_{d}}{k_{c}\left(L^{*}+\lambda_{d}\right)} \frac{m_{1} m_{2}}{m_{1}+m_{2}} e^{\frac{L^{*}}{\lambda_{d}}} f_{2}^{*} \tag{5.9}
\end{equation*}
$$

Note that $Q_{n}$ is a constant, it does not compensate for the distance error $\Delta x$. When implementing the feed-forward control, an estimated value of $f_{2}^{*}$ is required at the beginning of the control.

Note that to obtain an estimate $f_{2}^{*}$, measurements of both $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ are required at an instant. If the accuracy requirement of these measurements can be reduced, or the requirement for $f_{2}^{*}$ removed, then this charge control would be much simpler to implement.

### 5.2.2 Full-State Feedback Control \& Stability Analysis

The prior section determines the feed-forward charge product for a circular relative orbit by assuming a pure two-spacecraft system. This section develops the charge feedback component of the final control that stabilizes the shape errors.

Define a Lyapunov candidate function as

$$
\begin{equation*}
V=\frac{1}{2} p \Delta x^{2}+\frac{1}{2} \Delta \dot{x}^{2} \tag{5.10}
\end{equation*}
$$

Taking a time derivative of $V$ yields:

$$
\begin{align*}
\dot{V} & =\Delta \dot{x}(p \Delta x+\Delta \ddot{x}) \\
& =\Delta \dot{x}\left(k \Delta x+k_{c} \frac{Q}{L^{2}}\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right)\left(1+\frac{L}{\lambda_{d}}\right) e^{-\frac{L}{\lambda_{d}}}+f_{1}+f_{2}\right) \tag{5.11}
\end{align*}
$$

Ideally we would like to force $\dot{V}$ to be of the following negative semi-definite form:

$$
\begin{equation*}
\dot{V} \triangleq-d \Delta \dot{x}^{2} \tag{5.12}
\end{equation*}
$$

with $d>0$. Note that $\dot{V}$ is negative semi-definite because $V$ is a function of both $\Delta x$ and $\Delta \dot{x}$, but only $\Delta \dot{x}$ appears in $\dot{V}$. Studying the higher order derivatives of $V$ it can be shown that this control will be asymptotically stabilizing.

Substituting Eq. (5.11) into Eq. (5.12), and solving for the feedback charge product $\delta Q$, yields:

$$
\begin{align*}
\delta Q_{f} & =\frac{L^{2}}{k_{c}} \frac{m_{1} m_{2}}{m_{1}+m_{2}} \frac{\lambda_{d}}{L+\lambda_{d}} e^{-\frac{L}{\lambda_{d}}}\left(-p \Delta x-d \Delta \dot{x}-f_{1}-f_{2}\right)-Q_{n} \\
& =\frac{L^{2}}{k_{c}} \frac{m_{1} m_{2}}{m_{1}+m_{2}} \frac{\lambda_{d}}{L+\lambda_{d}} e^{-\frac{L}{\lambda_{d}}}\left(-p \Delta x-d \Delta \dot{x}-f_{1}-f_{2}+f_{2}^{*}\right) \tag{5.13}
\end{align*}
$$

Note that the $f_{2}^{*}$ term in the brackets comes from the feed-forward control $Q_{n}$. The usage of this term is to cancel out the function of the relative position vector $f_{2}$. However $f_{2}^{*}$ is a constant while $f_{2}$ is time varying, perfect canceling $f_{2}$ is not achievable. Because the $f_{1}$ function requires knowledge of the inertial position vectors of the two spacecraft, this feedback control in Eq. (5.13) is called full-state feedback control.

The full-state feedback control given by Eq. (5.13) ensures $\dot{V}$ to be negative semidefinite as shown in Eq. (5.12). Taking a second time derivative of $V$, yields

$$
\begin{equation*}
\ddot{V}=-2 d \Delta \dot{x} \Delta \ddot{x} \tag{5.14}
\end{equation*}
$$

When $\dot{V}=0, \Delta \dot{x}=0$, thus $\ddot{V}=0$. Taking a third time derivative of $V$, yields

$$
\begin{equation*}
\dddot{V}=-2 d \Delta \ddot{x}^{2}-2 d \Delta \dot{x} \Delta \dddot{x} \tag{5.15}
\end{equation*}
$$

When $\dot{V}=0, \dddot{V}=-2 d \Delta \ddot{x}^{2}<0$. Thus the system is asymptotically stable under the full-state feedback control in Eq. (5.13)

### 5.2.3 Partial-State Feedback Control \& Stability Analysis

The full-state feedback control given by Eqs. (5.9) and (5.13) developed in the previous section requires the measurement of the inertial and relative position vectors. If the measurement is accurate then the full-state feedback control is asymptotically stable. However, these position vectors are very difficult to measure accurately in a tight formation flying in GEO orbit with separation distance within 100m. This section studies the separation distance feedback control with the feedback components simplified to only require separation distance measurements:

$$
\begin{equation*}
\delta Q_{p}=\frac{L^{2}}{k_{c}} \frac{m_{1} m_{2}}{m_{1}+m_{2}} \frac{\lambda_{d}}{L+\lambda_{d}} e^{-\frac{L}{\lambda_{d}}}(-p \Delta x-d \Delta \dot{x}) \tag{5.16}
\end{equation*}
$$

The feed-forward part is given by Eq. (5.9). The feedback part $\delta Q_{p}$ in Eq. (5.16) is obtained by removing the $f_{1}$ function from $\delta Q_{f}$ in Eq. (5.13). It requires only the measurement of the separation distance which is easy to measure accurately. Substituting Eq. (5.9) and (5.16) into the EOM in Eq. (5.5) yields

$$
\begin{equation*}
\Delta \ddot{x}+d \Delta \dot{x}+p \Delta x=f_{1}+f_{2}-f_{2}^{*} \tag{5.17}
\end{equation*}
$$

Note that $f_{2}$ is a function of the relative position vector, it's time varying. Thus $f_{2}^{*}-f_{2}$ never stays at zero no matter what the guess of $f_{2}^{*}$ would be. In order to study this
error, let us start from the expression of $f_{2}$ :

$$
\begin{equation*}
f_{2}=\frac{1}{L}\|\dot{\boldsymbol{r}}\|^{2}\left(1-\cos ^{2} \angle(\boldsymbol{r}, \dot{\boldsymbol{r}})\right) \tag{5.18}
\end{equation*}
$$

It's beneficial if $f_{2}$ can be expressed in terms of the states $\Delta x$ and $\Delta \dot{x}$. In this way the Taylor series expansion can be utilized to linearize the function $f_{2}$ about the estimated value $f_{2}^{*}$. The following identities will be used in developing new expression of $f_{2}$ :

$$
\left\{\begin{align*}
\boldsymbol{r} & =L \hat{e}_{r}  \tag{5.19}\\
\dot{r} & =\dot{L} \hat{\boldsymbol{e}}_{r}+L \dot{\theta} \hat{\boldsymbol{e}}_{\theta}
\end{align*}\right.
$$

The cosine function in Eq. (5.18) is expressed by:

$$
\begin{equation*}
\cos \angle(\boldsymbol{r}, \dot{\boldsymbol{r}})=\frac{\boldsymbol{r} \cdot \dot{\boldsymbol{r}}}{\|\boldsymbol{r}\|\|\dot{\boldsymbol{r}}\|}=\frac{\dot{L}}{\sqrt{\dot{L}^{2}+(L \dot{\theta})^{2}}} \tag{5.20}
\end{equation*}
$$

For a fast spinning two-craft formation, the momentum is approximately conserved if the local gravity gradient torque can be ignored over the short-term (fraction of an orbit):

$$
\begin{equation*}
h=L^{2} \dot{\theta}=L^{* 2} \dot{\theta}^{*} \tag{5.21}
\end{equation*}
$$

where $L^{*}$ is the expected separation distance, $\dot{\theta}^{*}$ is the nominal spinning angular rate. Solving for $\dot{\theta}$ from Eq. (5.21) yields

$$
\begin{equation*}
\dot{\theta}=\frac{L^{* 2}}{L^{2}} \dot{\theta}^{*} \tag{5.22}
\end{equation*}
$$

Substituting Eq. (5.22) into Eq. (5.20) yields

$$
\begin{equation*}
\cos \angle(\boldsymbol{r}, \dot{\boldsymbol{r}})=\frac{\dot{L}}{\sqrt{\dot{L}^{2}+\left(\frac{L^{* 2}}{L} \dot{\theta}^{*}\right)^{2}}} \tag{5.23}
\end{equation*}
$$

Substituting Eqs. (5.19) and (5.23) into Eq. (5.18) yields

$$
\begin{equation*}
f_{2}=\frac{L^{* 4}}{L^{3}} \dot{\theta}^{2} \tag{5.24}
\end{equation*}
$$

In this expression only $L$ is a variable, other parameters are constants determined by the expected separation distance and nominal spinning rate. Thus $f_{2}$ is a function of $L$
by assuming a fast spinning two-craft formation compared to the orbit period. Taking a Taylor series expansion about the expected separation distance yields the first order relationship:

$$
\begin{equation*}
f_{2}(L)=f_{2}^{*}+\frac{d f_{2}}{d L} \Delta x=f_{2}^{*}-\frac{3 L^{* 4}}{L^{4}} \dot{\theta}^{2} \Delta x+O\left(\Delta x^{2}\right) \tag{5.25}
\end{equation*}
$$

Substituting Eq. (5.25) into the close-loop EOM in Eq. (5.17) yields

$$
\begin{equation*}
\Delta \ddot{x}+d \Delta \dot{x}+p \Delta x+\frac{3 h^{* 2}}{L^{4}} \Delta x=f_{1} \tag{5.26}
\end{equation*}
$$

where $h^{*}=L^{* 2} \dot{\theta}^{*}$ is the nominal momentum. Note that $f_{1}$ is a function of the inertial position vector. The next section will prove that the value of $f_{1}$ is very small for a formation in GEO orbit (the magnitude is up to $10^{-6} \mathrm{~m} / \mathrm{s}^{2}$ ), thus the influence of $f_{1}$ can be neglected for short-term stability discussions. Note that the close-loop dynamics in Eq. (5.26) is obtained by assuming the feed-forward part has perfect estimation $\hat{f}_{2}$ of the expected value $f_{2}^{*}$. If the estimation is not perfect, then there would exist a constant bias in the EOM. Denote the estimation error as

$$
\begin{equation*}
\delta f_{2}=f_{2}^{*}-\hat{f}_{2} \tag{5.27}
\end{equation*}
$$

then the EOM in Eq. (5.26) becomes

$$
\begin{equation*}
\Delta \ddot{x}+d \Delta \dot{x}+\left(p+\frac{3 h^{* 2}}{L^{4}}\right) \Delta x=\delta f_{2} \tag{5.28}
\end{equation*}
$$

The estimation error $\delta f_{2}$ acts as a constant perturbation to the system and may introduce bias or even destroy the stability of the system. To get rid of this constant error, this chapter uses an integral feedback term in the feedback control part:

$$
\begin{equation*}
\delta Q_{p 2}=\frac{L^{2}}{k_{c}} \frac{m_{1} m_{2}}{m_{1}+m_{2}} \frac{\lambda_{d}}{L+\lambda_{d}} e^{-\frac{L}{\lambda_{d}}}\left(-p \Delta x-d \Delta \dot{x}-k_{i} \int \Delta x\right) \tag{5.29}
\end{equation*}
$$

By assuming a fast spinning two-craft formation and ignoring the inertial position function $f_{1}$, the partial-state feedback control in Eq. (5.16) is proved to be stable. If there is an error of the estimated value of the expected $f_{2}^{*}$ function, there would be a
constant perturbation to the system that may introduce bias or instability factor. A new feedback control that includes an integral feedback is used to get rid of the constant bias. But the stability has not been analytically proved yet.

Schaub et. al. study the spinning 2-craft formation in Reference [13]. They prove that the 2-craft spinning Coulomb tether is passively stable in deep space. This chapter considers a different situation where the 2 -craft system is spinning in a GEO orbit. The gravitation forces are treated as extra disturbances. The stability is ensured for short term fast spin compared to the orbit rate. But long term stability is not ensured.

### 5.2.4 Boundaries Of The $f_{1}$ Function

The previous section develops an asymptotically stable full-state feedback controller and a stable partial-state feedback controller. The stability proof of the partialstate feedback controller assumes the influence of the inertial position function $f_{1}$ is neglectable. This section investigates the boundaries of the function $f_{2}$.


Figure 5.2: Geometry of the 2-craft system.

Let us start from the definition of $f_{1}$ :

$$
\begin{equation*}
f_{1}=\frac{G M}{R_{1}^{2}} \hat{\boldsymbol{o}}_{r 1} \cdot \hat{\boldsymbol{e}}_{r}-\frac{G M}{R_{2}^{2}} \hat{\boldsymbol{o}}_{r 2} \cdot \hat{\boldsymbol{e}}_{r} \tag{5.30}
\end{equation*}
$$

Because CFF considers formation with separation distance within 100 meters, $L$ is very small comparing with $R_{i}$. The following approximations have sufficient accuracy (the error is within $3 \times 10^{-5} \mathrm{~m}$ for the formation in GEO orbit):

$$
\begin{align*}
& R_{1}=R_{c}-\frac{1}{2} L \sin \alpha  \tag{5.31a}\\
& R_{2}=R_{c}+\frac{1}{2} L \sin \alpha \tag{5.31b}
\end{align*}
$$

where $\alpha$ is the angle between the unit vector $\hat{\boldsymbol{e}}_{r}$ and the local horizon plane as shown in Figure 5.2. $\alpha$ ranges within $[-90,90]^{\circ}$. From Figure 5.2, the unit vectors $\hat{\boldsymbol{o}}_{r 1}$ and $\hat{\boldsymbol{o}}_{r 2}$ can be expressed as

$$
\begin{align*}
& \hat{\boldsymbol{o}}_{r 1}=\frac{1}{R_{1}}\left(R_{c} \hat{\boldsymbol{o}}_{r c}-\frac{1}{2} L \hat{\boldsymbol{e}}_{r}\right)  \tag{5.32a}\\
& \hat{\boldsymbol{o}}_{r 2}=\frac{1}{R_{2}}\left(R_{c} \hat{\boldsymbol{o}}_{r c}+\frac{1}{2} L \hat{\boldsymbol{e}}_{r}\right) \tag{5.32b}
\end{align*}
$$

Substituting Eq. (5.31) and Eq. (5.32) into Eq. (5.30), yields:

$$
\begin{align*}
f_{1} & =G M\left(\frac{R_{c} \hat{\boldsymbol{o}}_{r c} \cdot \hat{\boldsymbol{e}}_{r}-0.5 L}{\left(R_{c}-0.5 L \sin \alpha\right)^{3}}-\frac{R_{c} \hat{\boldsymbol{o}}_{r c} \cdot \hat{\boldsymbol{e}}_{r}+0.5 L}{\left(R_{c}+0.5 L \sin \alpha\right)^{3}}\right) \\
& =G M\left(\frac{R_{c} \sin \alpha-0.5 L}{\left(R_{c}-0.5 L \sin \alpha\right)^{3}}-\frac{R_{c} \sin \alpha+0.5 L}{\left(R_{c}+0.5 L \sin \alpha\right)^{3}}\right) \tag{5.33}
\end{align*}
$$

Now the term $f_{1}$ has been expressed as a function of the center of mass (CM) radius $R_{c}$, the separation distance $L$ and the angle $\alpha$. Note that this chapter considers a short-distance formation in a GEO orbit, the CM radius can be approximated by the radius of the GEO orbit $R_{c}=4.2155 \times 10^{7} \mathrm{~m}$. The separation distance is within 100 meters, at the steady state it's close to the desired value. The angle $\alpha$ can not be controlled because Coulomb forces are internal forces in the formation and are not capable to directly control the inertial orientation of the formation. $\alpha$ is the most varying variable in the expression of $f_{1}$ in Eq. (5.33), and it's the only variable when the formation is at the steady state. The behavior of $f_{1}$ when $\alpha$ is changing should be identified.

Taking a partial derivative of $f_{1}$ with respect to (w.r.t.) $\alpha$, yields:

$$
\begin{align*}
\frac{\partial f_{1}}{\partial \alpha}=G M & \left\{\frac{R_{c} \cos \alpha}{\left(R_{c}-0.5 L \sin \alpha\right)^{3}}+\frac{1.5 L \cos \alpha\left(R_{c} \sin \alpha-0.5 L\right)}{\left(R_{c}-0.5 L \sin \alpha\right)^{4}}\right. \\
& \left.-\frac{R_{c} \cos \alpha}{\left(R_{c}+0.5 L \sin \alpha\right)^{3}}+\frac{1.5 L \cos \alpha\left(R_{c} \sin \alpha+0.5 L\right)}{\left(R_{c}+0.5 L \sin \alpha\right)^{4}}\right\} \\
=G M & \left\{\frac{1}{\left(R_{c}-0.5 L \sin \alpha\right)^{4}}\left(R_{c}^{2} \cos \alpha+R_{c} L \sin \alpha \cos \alpha-0.75 L^{2} \cos \alpha\right)\right. \\
& \left.-\frac{1}{\left(R_{c}+0.5 L \sin \alpha\right)^{4}}\left(R_{c}^{2} \cos \alpha-R_{c} L \sin \alpha \cos \alpha-0.75 L^{2} \cos \alpha\right)\right\} \tag{5.34}
\end{align*}
$$

The extrema occurs when $\frac{\partial f_{1}}{\partial \alpha}=0$. From Eq. (5.34), one obvious solution that makes the partial derivative be zero is $\cos \alpha=0$. When $\cos \alpha=0$ then $\sin \alpha= \pm 1$. Substituting $\sin \alpha= \pm 1$ into the expression of $f_{1}$ in Eq. (5.33), yields:

$$
\begin{equation*}
f_{1}^{(1)}=G M\left[\frac{1}{\left(R_{c}-0.5 L\right)^{2}}-\frac{1}{\left(R_{c}+0.5 L\right)^{2}}\right] \tag{5.35}
\end{equation*}
$$

Another solution that makes the partial derivative in Eq. (5.34) be zero is $\sin \alpha=0$. Substituting $\sin \alpha=0$ into Eq. (5.33), yields:

$$
\begin{equation*}
f_{1}^{(2)}=-\frac{G M L}{R_{c}^{3}} \tag{5.36}
\end{equation*}
$$

The following theorem proves that $f_{1}^{(1)}$ is the maximum of $f_{1}$, and $f_{1}^{(2)}$ is the minimum of $f_{1}$.

Theorem 7 Given a function of $\alpha$ defined by Eq. (5.33). Assume that $L$ is constant and $\alpha \in[-90,90]^{\circ}$. If $R_{c} \gg L$, then the maximum value occurs when $\cos \alpha=0$, the minimum value occurs when $\sin \alpha=0$. The maximum value is $f_{1}^{(1)}$ given by Eq. (5.35) and the minimum value is given by Eq. (5.36).

Proof The derivation from Eq. (5.34) to Eq. (5.36) has proved that $f_{1}^{(1)}$ and $f_{1}^{(2)}$ are two extrema of the function $f_{1}$. Further investigation is needed to show that these two extrema are the maximum and minimum point of the function. Taking a second order
partial derivative of $f_{1}$ w.r.t. $\alpha$, yields:

$$
\begin{align*}
\frac{\partial^{2} f_{1}}{\partial \alpha^{2}}= & G M\left\{\frac{1}{\left(R_{c}-0.5 L \sin \alpha\right)^{4}}\left(-R_{c}^{2} \sin \alpha+R_{c} L \cos ^{2} \alpha-R_{c} L \sin ^{2} \alpha+0.75 L^{2} \sin \alpha\right)\right. \\
& +\frac{2 L \cos \alpha}{\left(R_{c}-0.5 L \sin \alpha\right)^{5}}\left(R_{c}^{2} \cos \alpha+R_{c} L \sin \alpha \cos \alpha-0.75 L^{2} \cos \alpha\right) \\
& -\frac{1}{\left(R_{c}+0.5 L \sin \alpha\right)^{4}}\left(-R_{c}^{2} \sin \alpha-R_{c} L \cos ^{2} \alpha-R_{c} L \sin ^{2} \alpha+0.75 L^{2} \sin \alpha\right) \\
& \left.+\frac{2 L \cos \alpha}{\left(R_{c}+0.5 L \sin \alpha\right)^{5}}\left(R_{c}^{2} \cos \alpha-R_{c} L \sin \alpha \cos \alpha-0.75 L^{2} \cos \alpha\right)\right\} \tag{5.37}
\end{align*}
$$

When $\cos \alpha=0$ and $\sin \alpha=1, \alpha=\frac{\pi}{2}$. The second order partial derivative becomes:

$$
\left.\begin{array}{rl}
\left.\frac{\partial^{2} f_{1}}{\partial \alpha^{2}}\right|_{\alpha=90^{\circ}}= & G M\left(\frac{-R_{c}^{2}-R_{c} L+0.75 L^{2}}{\left(R_{c}-0.5 L\right)^{4}}-\frac{-R_{c}^{2}+R_{c} L+0.75 L^{2}}{\left(R_{c}+0.5 L\right)^{4}}\right) \\
= & G M
\end{array}\right)\left\{\left(-R_{c}^{2}+0.75 L^{2}\right)\left(\frac{1}{\left(R_{c}-0.5 L\right)^{4}}-\frac{1}{\left(R_{c}+0.5 L\right)^{4}}\right)\right\}
$$

Because $R_{c} \gg L,\left(-R_{c}^{2}+0.75 L^{2}\right)<0$. The following inequality is obvious:

$$
\begin{equation*}
\frac{1}{\left(R_{c}-0.5 L\right)^{4}}-\frac{1}{\left(R_{c}+0.5 L\right)^{4}}>0 \tag{5.39}
\end{equation*}
$$

So the value of the second partial derivative in Eq. (5.38) is negative:

$$
\begin{equation*}
\left.\frac{\partial^{2} f_{1}}{\partial \alpha^{2}}\right|_{\alpha=90^{\circ}}<0 \tag{5.40}
\end{equation*}
$$

When $\cos \alpha=0$ and $\sin \alpha=-1, \alpha=-\frac{\pi}{2}$. Then the second order partial derivative is:

$$
\begin{align*}
\left.\frac{\partial^{2} f_{1}}{\partial \alpha^{2}}\right|_{\alpha=-90^{\circ}}= & G M\left(\frac{R_{c}^{2}-R_{c} L-0.75 L^{2}}{\left(R_{c}+0.5 L\right)^{4}}-\frac{R_{c}^{2}+R_{c} L-0.75 L^{2}}{\left(R_{c}-0.5 L\right)^{4}}\right) \\
= & G M\left\{\left(R_{c}^{2}-0.75 L^{2}\right)\left(\frac{1}{\left(R_{c}+0.5 L\right)^{4}}-\frac{1}{\left(R_{c}-0.5 L\right)^{4}}\right)\right. \\
& \left.-R_{c} L\left(\frac{1}{\left(R_{c}+0.5 L\right)^{4}}+\frac{1}{\left(R_{c}-0.5 L\right)^{4}}\right)\right\} \tag{5.41}
\end{align*}
$$

Note that

$$
\begin{equation*}
\frac{1}{\left(R_{c}+0.5 L\right)^{4}}-\frac{1}{\left(R_{c}-0.5 L\right)^{4}}<0 \tag{5.42}
\end{equation*}
$$

So the partial derivative in Eq. (5.41) is negative:

$$
\begin{equation*}
\left.\frac{\partial^{2} f_{1}}{\partial \alpha^{2}}\right|_{\alpha=-90^{\circ}}<0 \tag{5.43}
\end{equation*}
$$

From the two results in Eqs. (5.40), (5.43), it can be concluded that $\cos \alpha=0$ is the maximum point of the $f_{1}$ function. This proves that $f_{1}^{(1)}$ is the maximum value of $f_{1}$.

When $\sin \alpha=0, \alpha=0$. The second order partial derivative is

$$
\begin{equation*}
\left.\frac{\partial^{2} f_{1}}{\partial \alpha^{2}}\right|_{\alpha=0}=G M\left\{\frac{2 R_{c} L}{R_{c}^{4}}+\frac{4 L}{R_{c}^{5}}\left(R_{c}^{2}-0.75 L^{2}\right)\right\} \tag{5.44}
\end{equation*}
$$

Clearly each term in Eq. (5.44) is positive, so the partial derivative in Eq. (5.44) is positive

$$
\begin{equation*}
\left.\frac{\partial^{2} f_{1}}{\partial \alpha^{2}}\right|_{\alpha=0}>0 \tag{5.45}
\end{equation*}
$$

This indicates that $f_{1}^{(2)}$ in Eq. (5.36) is the minimum value of the function $f_{1}$.
Theorem 7 proves that $f_{1}^{(1)}$ and $f_{1}^{(2)}$ are upper and lower bounds of the function $f_{1}$. Thus the value level of $f_{1}$ can be determined by these two boundaries. For a formation flying in a GEO orbit with separation distance within 100 m , the boundaries for $f_{1}^{(1)}$ and $f_{1}^{(2)}$ are determined:

$$
\begin{gather*}
f_{1}^{(1)} \leq 1.0646 \times 10^{-6} \mathrm{~m} / \mathrm{s}^{2}  \tag{5.46}\\
\left|f_{1}^{(2)}\right| \leq 5.3228 \times 10^{-7} \mathrm{~m} / \mathrm{s}^{2} \tag{5.47}
\end{gather*}
$$

Figure 5.3 shows the real values and boundaries of $f_{1}$ and $f_{2}$ in a simulation test. Figures $5.3(\mathrm{a})$ and $5.3(\mathrm{~b})$ show the distance error history and the control charge product history. After around 3000s the distance error settles down to be close to zero. Figure $5.3(\mathrm{c})$ shows the boundaries of $f_{1}$. Figure $5.3(\mathrm{~d})$ shows the true value and the estimation of the relative position feedback term $f_{2}$. Comparing with Figure 5.3(c), the magnitude of the function $f_{2}$ is 4 times in order greater than $f_{1}$. Thus the influence of the inertial position function $f_{1}$ can be ignored.


Figure 5.3: A simulation example to show the boundaries of $f_{1}$ and the history of $f_{2}$.

### 5.2.5 Comparison Of The Functions $f_{1}$ And $f_{2}$

The last section finds the upper and lower bounds of the function $f_{1}$ which is determined by the gravitational forces. A simulation case shows that the influence of $f_{1}$ is very small as compared to $f_{2}$. This section uses numerical sweeping to investigate in detail the magnitudes of $f_{1}$ and $f_{2}$ under different conditions. The results can help to determine whether the gravitation influence term $f_{1}$, which requires the inertial position feedback, can be ignored under a specific condition.

By the definitions of $f_{1}$ and $f_{2}$ in Eqs. (5.33) and (5.18) respectively, these two terms are varying with the separation distance and the relative speed. Figure 5.4 shows the the magnitudes of $f_{1}$ and $f_{2}$ by sweeping the value of the separation distance and the relative speed. Note that the values of $f_{1}$ and $f_{2}$ are calculated assuming the spinning


Figure 5.4: Comparison of $f_{1}$ and $f_{2}$ under different conditions.
two-craft system is in the nominal states, which indicates that the separation distance does not change and the relative velocity is perpendicular to the relative position vector.

Figure $5.4(\mathrm{a})$ shows the magnitudes of $f_{1}$ and $f_{2}$ when sweeping the separation distance. The relative speed magnitude is set to $1 \mathrm{~cm} / \mathrm{s}$. It shows that when $L<96, f_{2}$ is greater than $f_{1}$. When $L<20 \mathrm{~m}, f_{2}$ is at least one order greater than $f_{1}$. Figure 5.4(b) shows the magnitudes of $f_{1}$ and $f_{2}$ when sweeping the relative speed magnitude. It shows that when $v>0.41 \mathrm{~cm} / \mathrm{s}, f_{2}>f_{1}$. $f_{2}$ increases quadratically as the speed increases, $f_{1}$ does not change with respect to the relative speed.

Coulomb formation flying considers very tight formation with separation distances within 100 m . So from the above results, if the relative speed is at $\mathrm{cm} / \mathrm{s}$ level or above, the influence of $f_{2}$ dominates and $f_{1}$ can be ignored. Otherwise the influence of dropping the inertial feedback term $f_{1}$ maybe significant and needs to be considered carefully.

### 5.3 Numerical Simulations

A Lypunov-based nonlinear feedback control has been developed in the previous section. The control requires only the separation distance and rate feedback. It ignores the two position vectors' functions $f_{1}$ and $f_{2}$. The boundaries of the two functions
are investigated. In this section, several numerical simulations are used to test the performance of the controller and the behavior of the 2-craft formation.

The masses of the spacecraft are:

$$
\begin{equation*}
m_{1}=m_{2}=50 \mathrm{~kg} \tag{5.48}
\end{equation*}
$$

The mass of the Earth is $M=5.9742 \times 10^{24} \mathrm{~kg}$. The gravitational constant is $G=$ $6.67428 \times 10^{-11} \mathrm{~m}^{3} \mathrm{~kg}^{-1} \mathrm{~s}^{-2}$. Because the plasma shielding effect is strong at Low Earth Orbit (LEO), Coulomb formation flying considers formations in GEO or deep space. The initial position of the center of mass (CM) of the 2-craft system is set to be

$$
\begin{equation*}
\boldsymbol{R}_{c}\left(t_{0}\right)=\left[R_{c}, 0,0\right]^{T} \tag{5.49}
\end{equation*}
$$

where $R_{c}=42155000 \mathrm{~m}$ which is the radius of a GEO orbit. Note that the vector $\boldsymbol{R}_{c}\left(t_{0}\right)$ is expressed in the ECI frame. The initial positions of the two spacecraft are functions of $R_{c}\left(t_{0}\right)$ :

$$
\begin{equation*}
\boldsymbol{R}_{1}\left(t_{0}\right)=\boldsymbol{R}_{c}\left(t_{0}\right)-\frac{m_{2}}{m_{1}+m_{2}} \boldsymbol{r}\left(t_{0}\right), \quad \boldsymbol{R}_{2}\left(t_{0}\right)=\boldsymbol{R}_{c}\left(t_{0}\right)+\frac{m_{1}}{m_{1}+m_{2}} \boldsymbol{r}\left(t_{0}\right) \tag{5.50}
\end{equation*}
$$

where $\boldsymbol{r}\left(t_{0}\right)$ is the initial relative position vector expressed in the ECI frame. Note that the initial position of the $\mathrm{CM} \boldsymbol{R}_{c}\left(t_{0}\right)$ and the spacecraft masses $m_{1}$ and $m_{2}$ have been determined, the initial relative position vector $\boldsymbol{r}\left(t_{0}\right)$ determines the initial positions of the two spacecraft. The value of the relative position vector $\boldsymbol{r}\left(t_{0}\right)$ will be specified in the specific simulations cases.

The initial velocity of the CM of the two spacecraft system is defined as

$$
\begin{equation*}
\dot{\boldsymbol{R}}_{c}\left(t_{0}\right)=\left[0, v_{c}, 0\right]^{T} \mathrm{~m} / \mathrm{s} \tag{5.51}
\end{equation*}
$$

where $v_{c}=3070 \mathrm{~m} / \mathrm{s}$ is the nominal speed of a GEO orbit. Corresponding to the initial positions of the two spacecraft in Eq. (5.50), the initial velocities of the two spacecraft are given by:

$$
\begin{equation*}
\dot{\boldsymbol{R}}_{1}\left(t_{0}\right)=\dot{\boldsymbol{R}}_{c}\left(t_{0}\right)-\frac{m_{2}}{m_{1}+m_{2}} \dot{\boldsymbol{r}}\left(t_{0}\right), \quad \dot{\boldsymbol{R}}_{2}\left(t_{0}\right)=\dot{\boldsymbol{R}}_{c}\left(t_{0}\right)+\frac{m_{1}}{m_{1}+m_{2}} \dot{\boldsymbol{r}}\left(t_{0}\right) \tag{5.52}
\end{equation*}
$$

where $\dot{\boldsymbol{r}}\left(t_{0}\right)$ is the initial relative velocity. The value of $\dot{\boldsymbol{r}}\left(t_{0}\right)$ will be specified in the specific simulation cases as well.

### 5.3.1 Full-State Feedback Control Results

The full-state feedback control in Eq. (5.13) requires measurements of the inertial and relative position vectors. The benefit is that it's asymptotically stable. This simulation case shows the performance of the full-state feedback control. The initial relative position vector of the two spacecraft system is

$$
\begin{equation*}
\boldsymbol{r}\left(t_{0}\right)=[4,4,0]^{T} \mathrm{~m} \tag{5.53}
\end{equation*}
$$

The initial relative velocity is

$$
\dot{\boldsymbol{r}}\left(t_{0}\right)=\left[\begin{array}{lll}
0.02, & 0, & 0.02 \tag{5.54}
\end{array}\right]^{T} \mathrm{~m} / \mathrm{s}
$$

The expected separation distance is $L^{*}=4 \mathrm{~m}$. The Debye length is $\lambda_{d}=150 \mathrm{~m}$. The three controller coefficients are

$$
\begin{equation*}
p=1 \times 10^{-5} \mathrm{~s}^{-2}, \quad d=4 \times 10^{-3} \mathrm{~s}^{-1} \tag{5.55}
\end{equation*}
$$

Figure 5.5 shows the simulation results. Figure 5.5(a) shows the scenario as seen from the inertial frame centered at the CM of the two-craft system. The distance history in Figure 5.5(b) shows that the separation distance converges to the desired distance. Figure 5.5(c) shows the control charge product converges to the feed-forward charge product. Figure $5.5(\mathrm{~d})$ shows the magnitude of the Coulomb force. During the simulation the Coulomb force is within 10 mN .

### 5.3.2 Partial-State Feedback Simulation

This chapter develops two partial-state feedback control given by Eqs. (5.16) and (5.29). The control in Eq. (5.16) is stable assuming a fast spinning rate comparing to the GEO orbit rate. But when the estimation $\hat{f}_{2}$ is not equal to $f_{2}^{*}$, the separation


Figure 5.5: Full-state feedback control simulation.
distance would be biased to the expected distance. The control in Eq. (5.29) utilizes an integral feedback to compensate for the bias. But the stability is not proved.

The initial conditions and the control parameters are the same with the previous given by Eqs. (5.53)-(5.55). Figure 5.6 shows the simulation results using the feedback control in Eq. (5.16). In this case the feed-forward part has the perfect guess of the $f_{2}^{*}$ value. It can be seen that the distance converges to the expected distance and the charge product converges to the feed-forward charge product.

Figure 5.7 shows results of the same controller except that the estimation $\hat{f}_{2}$ is not equal to $f_{2}^{*}$. Figure 5.7 shows that there is a constant bias in the separation distance


Figure 5.6: Partial-state feedback control without integral feedback, with perfect estimation of $f_{2}^{*}$.


Figure 5.7: Partial-state feedback control without integral feedback, $\hat{f}_{2}=0.81 f_{2}^{*}$.
and the charge product. This control is stable, but it can not remove the constant bias.
Figure 5.8 shows simulation under the control in Eq. (5.29). The integral feedback coefficient is $k_{i}=1 \times 10^{-7} \mathrm{~S}^{-3}$. The integral feedback term removes the constant biases in the separation distance and the charge product. This shows the great advantage of the integral feedback control. But the stability of the feedback control with the integral feedback is not proved analytically.


Figure 5.8: Partial-state feedback control with integral feedback, $\hat{f}_{2}=0.81 f_{2}^{*}$.

### 5.4 Conclusion

This chapter investigates a two-craft Coulomb virtual structure control problem. A Lypunov-based full-state feedback control and a partial-state feedback control are developed. The full-state feedback control is asymptotically stable but it requires measurements of the inertial and relative position vectors which are difficult to obtain. The partial-state feedback control without integral feedback is stable assuming a fast spin rate. But the estimation error of the relative position function in the feed-forward part introduces a constant bias in the distance. An integral feedback term inserted into the partial-state feedback control removes the constant bias. But the nonlinear stability of the partial-state feedback control with the integral feedback is only shown with numerical simulations.

## CHAPTER 6

## ONE-DIMENSIONAL CONSTRAINT THREE-CRAFT COULOMB VIRTUAL STRUCTURE CONTROL

As the number of the spacecraft increases to three, the complexity of the charge control problem increases dramatically. Instead of studying the three-craft Coulomb virtual structure control in three-dimensional space directly, this chapter focuses on the 1-D restricted 3 -craft Coulomb virtual structure control to investigate charge implementability issues and charge saturation limitations. This 1-D constrained Coulomb structure control is a precursor for the more general study of the 3-D Coulomb structure control. Further, this 1-D constrained control is directly applicable to the 1-D non-conducting hover track control test bed which is under construction in the Automatic Vehicle Control (AVS) Lab in the Aerospace Engineering Sciences department at the University of Colorado at Boulder. The work in this chapter has been presented in Reference [30] and has been accepted to IEEE Transaction on Aerospace and Electronic System for publication.

### 6.1 Coulomb Virtual Structure Scenario

A Coulomb virtual structure is a cluster of spacecraft controlled by Coulomb forces to assume a certain shape. Because the shape is specified by the separation distances, the shape feedback control strategy uses the separation distances as the shape tracking error. The objectives of the controller are to make the separation distances
converge to desired values and to make the separation distance rates converge to zero (fixed nominal shape assumption).


Figure 6.1: One-dimensional Coulomb structure.

This chapter deals with an simple case, one-dimensional constraint 3-craft Coulomb structure. One-dinemsional constraint means the three spacecraft are lying on a line, and can only move on this line. Figure 6.1 shows the scenario of this case.

### 6.2 Charged Spacecraft Equations of Motion

The one-dimensional restricted Coulomb virtual structure simulates the motion of the test vehicles floating on a non-conducting hover track. The inertial positions of the three bodies are given through their inertial coordinates $x_{i}$. Without loss of generality, assume that $x_{1}<x_{2}<x_{3}$. Assume that the spacecraft are flying freely in space. In the scenario shown in Figure 6.1, assuming that the force acting from left to right to be positive, the inertial equations of motion of the charged bodies are given by

$$
\begin{align*}
& m_{1} \ddot{x}_{1}=k_{\mathrm{c}}\left[-\frac{Q_{12}}{\left(x_{2}-x_{1}\right)^{2}}-\frac{Q_{13}}{\left(x_{3}-x_{1}\right)^{2}}\right]  \tag{6.1}\\
& m_{2} \ddot{x}_{2}=k_{\mathrm{c}}\left[\frac{Q_{12}}{\left(x_{2}-x_{1}\right)^{2}}-\frac{Q_{23}}{\left(x_{3}-x_{2}\right)^{2}}\right]  \tag{6.2}\\
& m_{3} \ddot{x}_{3}=k_{\mathrm{c}}\left[\frac{Q_{13}}{\left(x_{3}-x_{1}\right)^{2}}+\frac{Q_{23}}{\left(x_{3}-x_{2}\right)^{2}}\right] \tag{6.3}
\end{align*}
$$

where $k_{\mathrm{c}}=8.99 \times 10^{9} \mathrm{C}^{-2} \cdot \mathrm{~N} \cdot \mathrm{~m}^{2}$ is the Coulomb constant, $Q_{i j}=q_{i} q_{j}$ is the charge product between the $i^{\text {th }}$ and $j^{\text {th }}$ craft. This product is introduced here because the
charges $q_{i}$ always appear in pairs $q_{i} q_{j}$ both in the dynamic equation and in the control formulation. This approach leads to the problem of physical feasibility in extracting individual charges $q_{i}$ from a given set of charge products $Q_{i j}$. This issue is addressed in the later sections (Sections 6.3.2.2 and 6.3.3.2). A charge feedback law is expected to control the relative motion of the three-body Coulomb structure and make the formation assume a specific shape defined through the separation distances.

Not all of the inertial $x_{i}$ states can be controlled independently. Because the spacecraft charges produce formation internal forces, the momentum of the Coulomb cluster must be conserved if there are no other external forces acting on it. As a result it is not possible to independently control all three inertial coordinates $x_{i}$ using only Coulomb forces. For the 1-D motion considered in this chapter, the conservation of the linear momentum imposes one constraint on the generalized coordinates $x_{1}, x_{2}$ and $x_{3}$. Thus, the motion of the three-body system only has two controlled degrees of freedom (DOF). The formation shape is defined through the two separation distances $\delta x_{12}$ and $\delta x_{23}$ as:

$$
\begin{equation*}
\delta x_{12}=x_{2}-x_{1}, \quad \delta x_{23}=x_{3}-x_{2} . \tag{6.4}
\end{equation*}
$$

The third distance $\delta x_{13}$ is determined by $\delta x_{13}=\delta x_{12}+\delta x_{23}$. To control the shape of the Coulomb structure is to drive $\left[\delta x_{12}, \delta x_{23}\right]^{T}$ to the desired constant values $\left[\delta x_{12}^{*}, \delta x_{23}^{*}\right]^{T}$ that yield a specific virtual structure shape. For the control development, let the system state vector $\boldsymbol{X}$ be defined as the relative distance tracking error:

$$
\boldsymbol{X}=\left[\begin{array}{l}
\Delta x_{12}  \tag{6.5}\\
\Delta x_{23}
\end{array}\right]=\left[\begin{array}{l}
\delta x_{12}-\delta x_{12}^{*} \\
\delta x_{23}-\delta x_{23}^{*}
\end{array}\right]
$$

This chapter only considers the shape control of the Coulomb structure, and does not attempt to control the formation cluster's center of mass motion. From the inertial equations of motion in Eqs. (6.1)-(6.3), using the definition of $\delta x_{i j}$, the separation
distance equations of motion are found as

$$
\begin{align*}
& \delta \ddot{x}_{12}=\ddot{x}_{2}-\ddot{x}_{1}=k_{\mathrm{c}}\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right) \frac{Q_{12}}{\delta x_{12}^{2}}-\frac{k_{\mathrm{c}}}{m_{2}} \frac{Q_{23}}{\delta x_{23}^{2}}+\frac{k_{\mathrm{c}}}{m_{1}} \frac{Q_{13}}{\delta x_{13}^{2}},  \tag{6.6}\\
& \delta \ddot{x}_{23}=\ddot{x}_{3}-\ddot{x}_{2}=-\frac{k_{\mathrm{c}}}{m_{2}} \frac{Q_{12}}{\delta x_{12}^{2}}+k_{\mathrm{c}}\left(\frac{1}{m_{2}}+\frac{1}{m_{3}}\right) \frac{Q_{23}}{\delta x_{23}^{2}}+\frac{k_{\mathrm{c}}}{m_{3}} \frac{Q_{13}}{\delta x_{13}^{2}} . \tag{6.7}
\end{align*}
$$

The formation kinetic energy $T$ is a convenient measure for constructing a Lyapunov function of the system and analyzing the stability of the equilibrium:

$$
\begin{equation*}
T=\frac{1}{2} \sum_{i=1}^{3} m_{i} \dot{x}_{i}^{2} . \tag{6.8}
\end{equation*}
$$

However, the control goal is to let the virtual structure assume a certain shape, which implies that the relative kinetic energy should be zero. Thus the inertial kinetic energy expression in Eq. (6.8) needs to be rewritten in terms of the relative coordinate rates $\delta \dot{x}_{12}$ and $\delta \dot{x}_{23}$. Taking a time derivative of Eq. (6.4) yields

$$
\begin{equation*}
\dot{x}_{1}=\dot{x}_{2}-\delta \dot{x}_{12}, \quad \dot{x}_{3}=\dot{x}_{2}+\delta \dot{x}_{23} \tag{6.9}
\end{equation*}
$$

Substituting Eq. (6.9) into Eq. (6.8) leads to

$$
\begin{equation*}
T=\frac{M}{2} \dot{x}_{2}^{2}+\frac{m_{1}}{2} \delta \dot{x}_{12}^{2}+\frac{m_{3}}{2} \delta \dot{x}_{23}^{2}+\dot{x}_{2}\left(m_{3} \delta \dot{x}_{23}-m_{1} \delta \dot{x}_{12}\right) \tag{6.10}
\end{equation*}
$$

where $M=\sum_{i=1}^{3} m_{i}$ is the total mass of the three spacecraft cluster. The expression of the total kinetic energy in Eq. (6.10) still contains an inertial rate variable $\dot{x}_{2}$ which cannot be controlled independently with Coulomb forces. One more step to express $\dot{x}_{2}$ in terms of $\delta \dot{x}_{i j}$ is needed.

Note that the Coulomb forces are internal forces in the Coulomb structure, by the assumption mentioned at the beginning that the spacecraft are flying freely in deep space, the following center of mass condition must be true:

$$
\begin{equation*}
m_{1} \dot{x}_{1}+m_{2} \dot{x}_{2}+m_{3} \dot{x}_{3}=M \dot{x}_{\mathrm{c}} \tag{6.11}
\end{equation*}
$$

where $x_{\mathrm{c}}$ is the inertial cluster center of mass coordinate. Utilizing Eq. (6.11), yields
the following equation:

$$
\begin{align*}
M \dot{x}_{2} & =M \dot{x}_{2}-m_{1} \dot{x}_{1}-m_{2} \dot{x}_{2}-m_{3} \dot{x}_{3}+M \dot{x}_{\mathrm{c}} \\
& =m_{1} \dot{x}_{2}-m_{1} \dot{x}_{1}+m_{2} \dot{x}_{2}-m_{2} \dot{x}_{2}+m_{3} \dot{x}_{2}-m_{3} \dot{x}_{3}+M \dot{x}_{\mathrm{c}} \\
& =m_{1} \delta \dot{x}_{12}-m_{3} \delta \dot{x}_{23}+M \dot{x}_{\mathrm{c}} \tag{6.12}
\end{align*}
$$

Thus $\dot{x}_{2}$ is expressed in terms of $\delta x_{i j}$ as:

$$
\begin{equation*}
\dot{x}_{2}=\frac{1}{M}\left(m_{1} \delta \dot{x}_{12}-m_{3} \delta \dot{x}_{23}\right)+\dot{x}_{\mathrm{c}} \tag{6.13}
\end{equation*}
$$

Substituting Eq. (6.13) into Eq. (6.10), yields

$$
\begin{equation*}
T=\frac{1}{2} \dot{\boldsymbol{X}}^{T}[M] \dot{\boldsymbol{X}}+\frac{M}{2} \dot{x}_{\mathrm{c}}^{2} \tag{6.14}
\end{equation*}
$$

where $[M]$ is the system mass matrix:

$$
[M]=\frac{1}{M}\left[\begin{array}{cc}
m_{1} m_{2}+m_{1} m_{3} & m_{1} m_{3}  \tag{6.15}\\
m_{1} m_{3} & m_{1} m_{3}+m_{2} m_{3}
\end{array}\right]
$$

Obviously, $[M]$ is a positive definite matrix. Finally, the kinetic energy $T_{\text {rel }}$ of the 3-craft cluster relative to the center of mass is given by

$$
\begin{equation*}
T_{\mathrm{rel}}=\frac{1}{2} \dot{\boldsymbol{X}}^{T}[M] \dot{\boldsymbol{X}} \tag{6.16}
\end{equation*}
$$

This energy expression directly reflects whether the virtual structure shape is changing its geometry with time.

### 6.3 Control Strategy

### 6.3.1 Shape Coordinate Equations of Motion

This section develops a continuous feedback control strategy that controls the 1-D 3-body formation to a certain desired shape. The desired shape is given by a vector of separation distances $\left[\delta x_{12}^{*}, \delta x_{23}^{*}\right]^{T}$, and it is assumed to be stationary (i.e. constant desired shape).

For notational convenience the $3 \times 1$ vector $\boldsymbol{\xi}$ is introduced as:

$$
\boldsymbol{\xi}=\left[\begin{array}{lll}
\frac{k_{\mathrm{c}} Q_{12}}{\delta x_{12}^{2}}, & \frac{k_{\mathrm{c}} Q_{23}}{\delta x_{23}^{2}}, & \frac{k_{\mathrm{c}} Q_{13}}{\delta x_{13}^{2}} \tag{6.17}
\end{array}\right]^{T}=k_{\mathrm{c}}[D] \boldsymbol{Q}
$$

where $[D]=\operatorname{diag}\left(\frac{1}{\delta x_{12}^{2}}, \frac{1}{\delta x_{23}^{2}}, \frac{1}{\delta x_{13}^{2}}\right)$ is a diagonal matrix, $\boldsymbol{Q}=\left[Q_{12}, Q_{23}, Q_{13}\right]^{T}$ is a vector of the charge products. The vector $\boldsymbol{Q}$ is also the control input of the Coulomb structure control system. Because the desired relative position coordinates are constants, the tracking error dynamics is expressed using $\boldsymbol{X}$ as

$$
\ddot{\boldsymbol{X}}=\underbrace{\left[\begin{array}{ccc}
\frac{1}{m_{1}}+\frac{1}{m_{2}} & -\frac{1}{m_{2}} & \frac{1}{m_{1}}  \tag{6.18}\\
-\frac{1}{m_{2}} & \frac{1}{m_{2}}+\frac{1}{m_{3}} & \frac{1}{m_{3}}
\end{array}\right]}_{[A]} \boldsymbol{\xi}=k_{\mathrm{c}}[A][D] \boldsymbol{Q}
$$

### 6.3.2 Formation Shape Control

The controller in this subsection is intended to make the formation attain a certain shape, which means both $\dot{\boldsymbol{X}}$ and $\boldsymbol{X}$ are driven to zero. For the time being the control development does not consider spacecraft charge saturation issues.

### 6.3.2.1 Minimum Norm Shape Stabilizing Control

Because the state vector $\boldsymbol{X}$ and the time derivative of the state vector $\dot{\boldsymbol{X}}$ are all expected to be zero, the Lyapunov function candidate here is defined as a quadratic function of $\boldsymbol{X}$ and $\dot{\boldsymbol{X}}$ as

$$
\begin{equation*}
V_{1}=\frac{1}{2} \dot{\boldsymbol{X}}^{T}[M] \dot{\boldsymbol{X}}+\frac{1}{2} \boldsymbol{X}^{T}[K] \boldsymbol{X} \tag{6.19}
\end{equation*}
$$

where $[K]$ is a $2 \times 2$ positive definite matrix. Because both $[M]$ and $[K]$ are positive definite, $V_{1}$ is a positive definite function of $\dot{\boldsymbol{X}}$ and $\boldsymbol{X}$. Note that the first term in $V_{1}$ is the relative kinetic energy $T_{\text {rel }}$ of the system.

Differentiating Eq. (6.19) with respect to time, and utilizing the shape error equations of motion in Eq. (6.18), yields

$$
\begin{equation*}
\dot{V}_{1}=\dot{\boldsymbol{X}}^{T}[K] \boldsymbol{X}+\dot{\boldsymbol{X}}^{T}[M] \ddot{\boldsymbol{X}}=\dot{\boldsymbol{X}}^{T}([K] \boldsymbol{X}+[M][A] \boldsymbol{\xi}) \tag{6.20}
\end{equation*}
$$

Denote $[C]=[M][A]$; it turns out to be a constant matrix with the following simple form:

$$
[C]=\left[\begin{array}{lll}
1 & 0 & 1  \tag{6.21}\\
0 & 1 & 1
\end{array}\right]
$$

Next the Lyapunov function rate $V_{1}$ is set to the negative semi-definite form

$$
\begin{equation*}
\dot{V}_{1}=-\dot{\boldsymbol{X}}^{T}[P] \dot{\boldsymbol{X}} \tag{6.22}
\end{equation*}
$$

where $[P]$ is a $2 \times 2$ positive definite matrix. $\dot{V}_{1}$ is negative semi-definite because $V_{1}$ is a function of both $\dot{\boldsymbol{X}}$ and $\boldsymbol{X}$, but only $\dot{\boldsymbol{X}}$ appears in Eq. (6.22).

Equating the actual $\dot{V}_{1}$ in Eq. (6.20) and the desired $\dot{V}_{1}$ in Eq. (6.22) leads to the following feedback control condition:

$$
\begin{equation*}
[C] \boldsymbol{\xi}=-[K] \boldsymbol{X}-[P] \dot{\boldsymbol{X}} \tag{6.23}
\end{equation*}
$$

Solving Eq. (6.23) for $\boldsymbol{\xi}$ yields the charge product vector that stabilizes the system. Because $[C]$ only has rank 2 , there is an infinite number of solutions for $\boldsymbol{\xi}$ in Eq. (6.23). Let $\hat{\boldsymbol{\xi}}$ be the minimum norm solution to Eq. (6.23):

$$
\begin{equation*}
\hat{\boldsymbol{\xi}}=-[C]^{\dagger}([K] \boldsymbol{X}+[P] \dot{\boldsymbol{X}}) \tag{6.24}
\end{equation*}
$$

where $[C]^{\dagger}=[C]^{T}\left([C][C]^{T}\right)^{-1}$ is the minimum norm pseudo-inverse of matrix $[C]$. The hat symbol above the vector $\boldsymbol{\xi}$ means that $\hat{\boldsymbol{\xi}}$ given by Eq. (6.24) is the minimum norm solution among the general solutions to Eq. (6.23); and $\hat{\boldsymbol{\xi}}$ is not the final solution of $\boldsymbol{\xi}$ that will be used in the control. Note that $\hat{\boldsymbol{\xi}}$ in Eq. (6.24) minimizes the norm of the charge product vector while satisfying Eq. (6.23), but not the charge inputs $q_{i}$ of the control.

### 6.3.2 2 Spacecraft Charge Computation Issues

After obtaining a solution $\boldsymbol{\xi}$ to Eq. (6.23), the charge product vector is given by

$$
\begin{equation*}
\boldsymbol{Q}=\frac{1}{k_{\mathrm{c}}}[D]^{-1} \boldsymbol{\xi} \tag{6.25}
\end{equation*}
$$

The individual charges $q_{i}$ are finally calculated through the algorithm [4]

$$
\begin{align*}
& q_{1}=\sqrt{\frac{Q_{12} Q_{13}}{Q_{23}}}  \tag{6.26a}\\
& q_{2}=\operatorname{sign}\left(Q_{12}\right) \frac{Q_{12}}{q_{1}}  \tag{6.26b}\\
& q_{3}=\operatorname{sign}\left(Q_{13}\right) \frac{Q_{13}}{q_{1}} \tag{6.26c}
\end{align*}
$$

Note that a singularity occurs if $\xi_{1} \cdot \xi_{2} \cdot \xi_{3}=0$. When one or two elements of $\boldsymbol{\xi}$ equal zero, this singularity can be avoided by performing a search routine in the null space of the $[C]$ matrix which will be discussed in the following several paragraphs. The remaining case is that $\boldsymbol{\xi}=\mathbf{0}$ which indicates that $q_{1}=q_{2}=q_{3}=0$. This state occurs only either when $\boldsymbol{X}=0$ and $\dot{\boldsymbol{X}}=0$, which means the system has reached the desired state, or due to $(-[K] \boldsymbol{X}-[P] \dot{\boldsymbol{X}})$ being zero.

Now consider general cases where $\xi_{1} \cdot \xi_{2} \cdot \xi_{3} \neq 0$. Note that $\xi_{1} \cdot \xi_{2} \cdot \xi_{3}<0$ yields imaginary values of $q_{i}$ [4]. Since charges must always be real numbers, $\xi_{1} \cdot \xi_{2} \cdot \xi_{3}<0$ is not an implementable solution. This is a fundamental issue with developing any charge feedback law.

Eq. (6.24) provides the minimum norm solution $\hat{\boldsymbol{\xi}}$ of $\boldsymbol{\xi}$ to Eq. (6.23). There is an infinite number of solutions that satisfy Eq. (6.23) since the matrix [ $C$ ] is a $2 \times 3$ matrix. Using the null space of $[C]$, all possible $\boldsymbol{\xi}$ values that satisfy Eq. (6.23) are parameterized as

$$
\boldsymbol{\xi}=\left(\begin{array}{l}
\xi_{1}  \tag{6.27}\\
\xi_{2} \\
\xi_{3}
\end{array}\right)=\hat{\boldsymbol{\xi}}+\gamma\left(\begin{array}{c}
-1 \\
-1 \\
1
\end{array}\right)
$$

where the parameter $\gamma$ can be any real number. The control problem is reformulated to determine a parameter $\gamma$ that satisfies the implementability constraint:

$$
\begin{equation*}
f(\gamma)=\xi_{1} \cdot \xi_{2} \cdot \xi_{3}=\left(\hat{\xi}_{1}-\gamma\right)\left(\hat{\xi}_{2}-\gamma\right)\left(\hat{\xi}_{3}+\gamma\right)>0 \tag{6.28}
\end{equation*}
$$

This inequality constraint guarantees that the charges $q_{i}$ are real, and also ensures that
the singularity case $\xi_{1} \cdot \xi_{2} \cdot \xi_{3}=0$ does not occur. Because $f(\gamma)$ is a third order function, there always exists real numbers of parameter $\gamma$ that satisfy the inequality in Eq. (6.28).

### 6.3.2.3 Charge Minimization Routine

Any real value of parameter $\gamma$ that satisfies the inequality in Eq. (6.28) makes the solution physically implementable with real charge $q_{i}$ solutions. In fact, the null space of the input matrix $[C]$ can be used to charge up the vehicles without causing any relative motion to occur. The $\hat{\boldsymbol{\xi}}$ vector is found such that the norm of the vector $\boldsymbol{\xi}$ is minimized. However, this does not correspond to the solution that the spacecraft charges $q_{i}$ are minimized. Define a charge cost function $J(\gamma)$ as

$$
\begin{equation*}
J(\gamma)=\sum_{i=1}^{3} q_{i}^{2} \tag{6.29}
\end{equation*}
$$

The solution $\boldsymbol{\xi}$ that minimizes spacecraft charges $q_{i}$ corresponds to a particular $\gamma_{m}$ that satisfies the inequality constraint in Eq. (6.28), and at the same time minimizes the charge cost function $J(\gamma)$.

Consider the constraint inequality in Eq. (6.28), where ( $\hat{\xi}_{1}, \hat{\xi}_{2}, \hat{\xi}_{3}$ ) are given by Eq. (6.24). There are three real roots for the equation $f(\gamma)=0$ which are $\left(\hat{\xi}_{1}, \hat{\xi}_{2},-\hat{\xi}_{3}\right)$. We rearrange the roots in a descending order and denote them as $\left(\gamma_{1}, \gamma_{2}, \gamma_{3}\right)$, where $\gamma_{1} \geq \gamma_{2} \geq \gamma_{3}$. The solution to the constraint in Eq. (6.28) turns out to be $\gamma>\gamma_{1}$ or $\gamma_{3}<\gamma<\gamma_{2}$. If $\gamma_{2}=\gamma_{3}$, then the solution is simply $\gamma>\gamma_{1}$. Figure 6.2(a) shows a numerical example of $f(\gamma)$ and $\left(\gamma_{1}, \gamma_{2}, \gamma_{3}\right)$.

Thus a charge minimizing routine is introduced to search for the parameter $\gamma_{m}$ within the two open intervals $\left(\gamma_{1}, \infty\right)$ and $\left(\gamma_{3}, \gamma_{2}\right)$. The numerical search algorithm used in this chapter is the secant method shown in Figure 6.3.

Once $\gamma_{m}$ is obtained, the solution that minimizes the norm of the charge vector $\left(q_{1}, q_{2}, q_{3}\right)$ is achieved, and of course it's also implementable. Figure 6.2 shows an example of the search result at one instant, where $\gamma_{\mathrm{m} 1}$ and $\gamma_{\mathrm{m} 2}$ are two local minimization

(a) $f(\gamma)$ and $\left(\gamma_{1}, \gamma_{2}, \gamma_{3}\right)$.

(b) $\gamma_{m}$ search result.

Figure 6.2: Illustration of $\gamma_{m}$ search routine.


Figure 6.3: Illustration of $\gamma_{m}$ search routine.
points.
Notice that generally there are two eligible intervals in the search routine. Sometimes this may introduce chatter because $\gamma_{m}$ switches between $\gamma_{\mathrm{m} 1}$ and $\gamma_{\mathrm{m} 2}$ when $J\left(\gamma_{\mathrm{m} 1}\right)$
and $J\left(\gamma_{\mathrm{m} 2}\right)$ are very close. To reduce the chatter of the charge history, one approach is to change the criteria for $\gamma_{\mathrm{m}}$ to switch between the two intervals. If $\gamma_{\mathrm{m}}(i)=\gamma_{\mathrm{m} 1}(i)$, then $\gamma_{\mathrm{m}}(i+1)=\gamma_{\mathrm{m} 2}(i+1)$ if and only if $J\left(\gamma_{\mathrm{m} 2}\right)<\alpha J\left(\gamma_{\mathrm{m} 1}\right)$, where $0<\alpha \leq 1$. Or in words, the charge solutions are only switched to the alternate set if the change in the cost function is sufficiently large.

### 6.3.3 Formation Shape Rate Regulation

This subsection develops a regulator that arrests the relative motion of the formation by driving $\dot{\boldsymbol{X}}$ to zero. After presenting a saturated stabilizing control strategy, a method to obtain implementable spacecraft charges $q_{i}$ is introduced.

### 6.3.3.1 Saturated Regulator

Because the purpose of the control is different from that of the shape control presented in section 6.3.2, a new Lyapunov function is introduced catering to the new demand. The regulator is intended to stop any relative motion of the formation, so the new Lyapunov function candidate $V_{2}$ is defined in terms of the relative velocity vector in a quadratic, positive definite form:

$$
\begin{equation*}
V_{2}=T_{\text {rel }}=\frac{1}{2} \dot{\boldsymbol{X}}^{T}[M] \dot{\boldsymbol{X}} \tag{6.30}
\end{equation*}
$$

Taking time derivative of $V_{2}$, and using the tracking error dynamics in Eq. (6.18), yields

$$
\begin{equation*}
\dot{V}_{2}=\dot{\boldsymbol{X}}^{T}[M] \ddot{\boldsymbol{X}}=k_{\mathrm{c}} \dot{\boldsymbol{X}}^{T}[C][D] \boldsymbol{Q} \tag{6.31}
\end{equation*}
$$

The saturated control strategy attempts to drive the rates $\dot{\boldsymbol{X}}$ to zero as quickly as possible, leading to a Lyapunov optimal control development [31]. Here the spacecraft charges are always held at the maximum magnitude. The control algorithm will need to determine the required signs of the spacecraft charges. The charge product vector $\boldsymbol{Q}$
is expressed as

$$
\boldsymbol{Q}=\left[\begin{array}{ccc}
Q_{12 \mathrm{~m}} & 0 & 0  \tag{6.32}\\
0 & Q_{23 \mathrm{~m}} & 0 \\
0 & 0 & Q_{13 \mathrm{~m}}
\end{array}\right]\left[\begin{array}{c}
s_{1} \\
s_{2} \\
s_{3}
\end{array}\right]=\left[Q_{\mathrm{m}}\right] s
$$

where $Q_{i j \mathrm{~m}}=q_{i \mathrm{~m}} q_{j \mathrm{~m}}$ is the product of the charge saturation limits of the $i^{\text {th }}$ and $j^{\text {th }}$ spacecraft. The vector $s=\operatorname{sign}(\boldsymbol{Q})$ is a $3 \times 1$ sign vector with the components being $\pm 1$ or zero. The matrix $\left[Q_{\mathrm{m}}\right]$ is a constant matrix determined by charge limitations of the spacecraft. Because $\left[Q_{\mathrm{m}}\right]$ is constant for a given 3-body Coulomb structure, the charge product $Q$ is determined only by $s$. Thus the vector $s$ is actually the essential variable that determines the saturated regulator. The Lyapunov function rate is rewritten as

$$
\begin{equation*}
\dot{V}_{2}=k_{\mathrm{c}} \dot{\boldsymbol{X}}^{T}[C][D]\left[Q_{\mathrm{m}}\right] \boldsymbol{s} \tag{6.33}
\end{equation*}
$$

To guarantee stability, the Lyapunov rate function $\dot{V}$ is set to be a negative semi definite function as

$$
\begin{equation*}
\dot{V}_{2}=-\dot{\boldsymbol{X}}^{T}[P] \dot{\boldsymbol{X}} \tag{6.34}
\end{equation*}
$$

where $[P]$ is a $2 \times 2$ positive matrix. Note that Eq. (6.34) has the same form as Eq. (6.22).
Substituting Eq. (6.34) into Eq. (6.33) provides an equation to solve for $s$. At first, let us treat $s$ as a general vector instead of a sign vector. A sign vector can be obtained by evaluating the signs of the elements in $s$. Note that $[C]$ is a $2 \times 3$ matrix, thus there is an infinite number of solutions for $\boldsymbol{s}$ after equating Eq. (6.34) and Eq. (6.33). Using the pseudo-inverse of matrix $[C]$, leads to the minimum norm solution $\tilde{\boldsymbol{s}}$ (the tilde symbol means $\tilde{s}$ is not a sign vector) :

$$
\begin{equation*}
\tilde{\boldsymbol{s}}=-\frac{1}{k_{\mathrm{c}}}\left[Q_{\mathrm{m}}\right]^{-1}[D]^{-1}[C]^{\dagger}[P] \dot{\boldsymbol{X}} \tag{6.35}
\end{equation*}
$$

Define a sign vector $s$ as:

$$
\begin{equation*}
s=\operatorname{sign}(\tilde{\boldsymbol{s}})=-\operatorname{sign}\left(\frac{1}{k_{\mathrm{c}}}\left[Q_{\mathrm{m}}\right]^{-1}[D]^{-1}[C]^{\dagger}[P] \dot{\boldsymbol{X}}\right) \tag{6.36}
\end{equation*}
$$

Here $s$ is a sign vector, but it may be un-implementable. This problem will be discussed following this subsection. Substituting $\boldsymbol{s}$ in Eq. (6.36) into charge vector $\boldsymbol{Q}$ in Eq. (6.32) constructs a saturated charge product control law:

$$
\begin{equation*}
\boldsymbol{Q}=\left[Q_{\mathrm{m}}\right] \boldsymbol{s}=-\left[Q_{\mathrm{m}}\right] \operatorname{sign}\left(\frac{1}{k_{\mathrm{c}}}\left[Q_{\mathrm{m}}\right]^{-1}[D]^{-1}[C]^{\dagger}[P] \dot{\boldsymbol{X}}\right) . \tag{6.37}
\end{equation*}
$$

The resulting actual Lyapunov function rate should be investigated, because after taking the sign function of $\tilde{\boldsymbol{s}}$, the actual Lyapunov function rate is different from the nominal one in Eq. (6.34). Substituting the actual charge product in Eq. (6.32) into Eq. (6.31), yields

$$
\begin{equation*}
\dot{V}_{2}=k_{\mathrm{c}} \dot{\boldsymbol{X}}^{T}[C][D]\left[Q_{\mathrm{m}}\right] \boldsymbol{s}=k_{\mathrm{c}} \dot{\boldsymbol{X}}^{T}[C][D]\left[Q_{\mathrm{m}}\right] \operatorname{sign}(\tilde{\boldsymbol{s}}) \tag{6.38}
\end{equation*}
$$

Note that the sign function can be deemed as a rescaling of the magnitude of a number, a scale matrix is introduced:

$$
\begin{equation*}
[E]=\operatorname{diag}\left(a_{1}, a_{2}, a_{3}\right), \tag{6.39}
\end{equation*}
$$

where $a_{i}$ is defined as

$$
a_{i}=\left\{\begin{array}{ll}
\frac{1}{\left\|\tilde{s}_{i}\right\|}, & \text { if } \quad \tilde{s}_{i} \neq 0  \tag{6.40}\\
0, & \text { if } \quad \tilde{s}_{i}=0
\end{array} .\right.
$$

Thus $\boldsymbol{s}$ can be rewritten as

$$
\begin{equation*}
s=[E] \tilde{s} . \tag{6.41}
\end{equation*}
$$

Substituting Eq. (6.41) into Eq. (6.38), and using Eq. (6.35) yields

$$
\begin{align*}
\dot{V}_{2} & =k_{\mathrm{c}} \dot{\boldsymbol{X}}^{T}[C][D]\left[Q_{\mathrm{m}}\right][E] \tilde{\boldsymbol{s}} \\
& =-\dot{\boldsymbol{X}}^{T} \underbrace{[C][D]\left[Q_{\mathrm{m}}\right][E]\left[Q_{\mathrm{m}}\right]^{-1}[D]^{-1}[C]^{\dagger}[P]}_{[F]} \dot{\boldsymbol{X}} \tag{6.42}
\end{align*}
$$

Without loss of generality, set the positive definite matrix $[P]$ introduced in Eq. (6.34) to be a diagonal matrix:

$$
[P]=\left[\begin{array}{cc}
p_{1} & 0  \tag{6.43}\\
0 & p_{2}
\end{array}\right]
$$

Utilizing previous definitions of matrices $[C],[D],\left[Q_{m}\right],[E]$, and $[P]$, the matrix $[F]$ is expanded as:

$$
[F]=\frac{1}{3}\left[\begin{array}{ll}
p_{1}\left(2 a_{1}+a_{3}\right) & p_{2}\left(-a_{1}+a_{3}\right)  \tag{6.44}\\
p_{1}\left(-a_{2}+a_{3}\right) & p_{2}\left(2 a_{2}+a_{3}\right)
\end{array}\right]
$$

From the condition $p_{i}>0$, it can be verified that the matrix $[F]$ is positive definite if $a_{i}>0$ and it is positive semi definite if $a_{i} \geq 0$. By the definition of matrix $[E], a_{i} \geq 0$. So the matrix $[F]$ is positive semi definite. The sign of the actual Lyapunov function rate is then determined:

$$
\begin{equation*}
\dot{V}_{2}=-\dot{\boldsymbol{X}}^{T}[F] \dot{\boldsymbol{X}} \leq 0 \tag{6.45}
\end{equation*}
$$

Thus the saturated control law in Eq. (6.36) is globally stable. But it's not asymptotically stable because the matrix $[F]$ can be zero if the states $\boldsymbol{X}$ grow infinitely large.

### 6.3.3.2 Implementable Saturated Control

The saturated charge product control in Eq. (6.37) provides a globally stable control that stops the relative motion of the formation. But this formula does not ensure physical implementability of the charge products. Similar to the shape controller's design, an implementable sign vector $\boldsymbol{s}=\left[s_{1}, s_{2}, s_{3}\right]$ must satisfy:

$$
\begin{equation*}
s_{1} \cdot s_{2} \cdot s_{3}>0 \tag{6.46}
\end{equation*}
$$

Unlike the case in the shape control design, the saturated regulator should be dealt with care because the sign function (or the matrix $[E]$ ) scales everything inside its argument. Note that the matrix $[E]$ is also varying with its argument. The previous approach that explores the null space of a certain matrix does not easily work out because of the rescaling of the matrix $[E]$, and the coupling of the matrix $[E]$ with the sign function's argument.

Note that in designing the stabilizing saturated control using Lyapunov stability theory, the stability property is achieved by setting the Lyapunov function rate to be
negative semi-definite. This is ensured by the positive-definite property of the $2 \times 2$ matrix $[P]$. In most cases, this matrix is constant because usually it's unnecessary to change the value of the matrix $[P]$ and a constant $[P]$ matrix may result in a better convergence property of the system. Because the saturated control in Eq. (6.37) is globally stable but not asymptotically stable, changing the matrix $[P]$ won't sacrifice convergence property of the system. Since the matrix $[P]$ is only required to be positivedefinite to guarantee the stability of the system, there exists a flexibility in choosing $[P]$.

Without loss of generality, set the matrix $[P]$ to be diagonal: $[P]=\operatorname{diag}\left(p_{1}, p_{2}\right)$. For $[P]$ to be positive-definite, the parameters $p_{1}$ and $p_{2}$ must be positive. Let $p_{1}$ and $p_{2}$ be constants. To set up a varying matrix $[P]$, a variable parameter $\tau$ is introduced to rewrite the matrix $[P]$ as

$$
[P]=\left[\begin{array}{cc}
p_{1} & 0  \tag{6.47}\\
0 & \tau p_{2}
\end{array}\right]
$$

here $\tau>0$ should be positive to ensure $[P]$ to be positive-definite. Note that because the matrices $\left[Q_{m}\right]$ and $[D]$ are all positive definite and diagonal, the sign vector in Eq. (6.36) can be simplified as

$$
\begin{equation*}
s=-\operatorname{sign}\left([C]^{\dagger}[P] \dot{\boldsymbol{X}}\right) \tag{6.48}
\end{equation*}
$$

Substituting the values of the matrices $[C]^{\dagger}$ and $[P]$ into Eq. (6.48), the vector $s$ is expanded as

$$
s=-\operatorname{sign}\left(\frac{1}{3}\left[\begin{array}{c}
2 p_{1} \dot{x}_{12}-\tau p_{2} \dot{x}_{23}  \tag{6.49}\\
-p_{1} \dot{x}_{12}+2 \tau p_{2} \dot{x}_{23} \\
p_{1} \dot{x}_{12}+\tau p_{2} \dot{x}_{23}
\end{array}\right]\right)
$$

For the sign vector $s$ to result in an implementable control, the vector inside the sign function must satisfy

$$
\begin{equation*}
\left(2 p_{1} \dot{x}_{12}-\tau p_{2} \dot{x}_{23}\right)\left(-p_{1} \dot{x}_{12}+2 \tau p_{2} \dot{x}_{23}\right)\left(p_{1} \dot{x}_{12}+\tau p_{2} \dot{x}_{23}\right)<0 \tag{6.50}
\end{equation*}
$$

transform the inequality in Eq. (6.50) to be:

$$
\begin{equation*}
g(\tau)=\left(p_{2} \dot{x}_{23} \tau-2 p_{1} \dot{x}_{12}\right)\left(2 p_{2} \dot{x}_{23} \tau-p_{1} \dot{x}_{12}\right) \cdot\left(p_{2} \dot{x}_{23} \tau+p_{1} \dot{x}_{12}\right)>0 \tag{6.51}
\end{equation*}
$$

Now the logic is clear that to find an implementable control by varying the matrix $[P]$ is to find a parameter $\tau>0$ that satisfies the inequality $g(\tau)>0$. Next the existence of a solution is verified. When $\dot{x}_{23}>0$, the inequality in Eq. (6.51) can be transformed to

$$
\begin{equation*}
h(\tau)=(\tau-\underbrace{\frac{2 p_{1} \dot{x}_{12}}{p_{2} \dot{x}_{23}}}_{b_{1}})(\tau-\underbrace{\frac{p_{1} \dot{x}_{12}}{2 p_{2} \dot{x}_{23}}}_{b_{2}})(\tau+\underbrace{\frac{p_{1} \dot{x}_{12}}{p_{2} \dot{x}_{23}}}_{-b_{3}})>0 \tag{6.52}
\end{equation*}
$$

Note that $h(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$. There always exists $\tau>0$ such that $h(\tau)>0$.
If $\dot{x}_{23}<0$, the inequality in Eq. (6.52) changes to be

$$
\begin{equation*}
h(\tau)<0 \tag{6.53}
\end{equation*}
$$

Note that $\left(b_{1}, b_{2}, b_{3}\right)$ are the three roots to the equation $h(\tau)=0$, and they share the simple relation $\operatorname{sign}\left(b_{1}\right)=\operatorname{sign}\left(b_{2}\right)=-\operatorname{sign}\left(b_{3}\right)$. When $b_{1}, b_{2}>0$ and $b_{3}<0$, then any $\tau \in\left(b_{2}, b_{1}\right)$ satisfies $h(\tau)<0$. If $b_{1}, b_{2}<0$ and $b_{3}>0$, in this case any $\tau \in\left(0, b_{3}\right)$ satisfies $h(\tau)<0$.

Note that $\dot{x}_{12}=0$ or $\dot{x}_{23}=0$ are transient states, unless $\dot{\boldsymbol{X}}=0$ which means the relative motion has been arrested. Thus there always exists $\tau>0$ that results in an implementable control.

### 6.4 Domains of Convergence

So far a two-stage control strategy has been presented to control the 1-D Coulomb formation. At first a saturated charge control is used to stop the relative motion of the 3 spacecraft. After the relative motion converges to zero, the formation shape control is activated to make the spacecraft to form a certain shape defined by the provided distances.

As mentioned before, the saturated charge control in Eq. (6.37) is globally stable, but not asymptotically stable. Under some initial conditions, such as the three spacecraft flying apart too fast, the relative motion cannot be arrested. This section is going to determine the domains of the initial conditions that result in stabilizable motions.

### 6.4.1 Convergence Criterion For Symmetric Relative Motion

In setting up experiments on hover track test bed, it's needed to know whether a configuration of the 1-D Coulomb structure can be stabilized. This section tries to find analytical conditions for stabilizable symmetric motions. Even though the symmetric motion is a special case for the 1-D Coulomb formation, it can be implemented in the hover track test bed.

Here the phrase "symmetric relative motion" means the distances between any two adjacent spacecraft are always equal to each other, and the adjacent distance rates are also equal. That is

$$
\begin{equation*}
\delta x_{12}=\delta x_{23}, \quad \delta \dot{x}_{12}=\delta \dot{x}_{23} \tag{6.54}
\end{equation*}
$$

Corresponding to this situation, the masses and charge limits of each body should all be equal, $m_{1}=m_{2}=m_{3}=m, q_{1 \max }=q_{2 \max }=q_{3 \max }=q_{\max }$. In this case the description of the motion can be greatly simplified. This simplified case will provide analytical insight into the specific instance when the saturated charge control is able to arrest any relative expansion.

For the 1-D Coulomb formation, the most likely scenario which could result in an unarrestable motion is that three spacecraft are departing from each other. That is $\delta \dot{x}_{12}>0$ and $\delta \dot{x}_{23}>0$. The following discussion deals with this "worst" case to find the criterion for the arrestable motions. The unarrestable motion happens when the center spacecraft attracts the two other spacecraft, but the distance rate vector $\dot{\boldsymbol{X}}$ still does
not decrease to zero. In this case the charges of the 3 spacecraft are

$$
\begin{equation*}
q_{1}=q_{3}= \pm q_{\max }, \quad q_{2}=\mp q_{\max } \tag{6.55}
\end{equation*}
$$

Chapter 3 presents an analytical way to find the criteria for the avoidance of a potential collision between 2 charged craft. It assumes that the charge product is constant, thus the trajectory of the 2-body motion is a conic section. Utilizing the methodology from the gravitational 2-body problem (2BP), the criteria is found through calculating the periapsis radius which is the closest distance between the 2 spacecraft in the conic section trajectory.

Motivated by this analytical approach to solve the 2-body Coulomb forced motion, another concept from the traditional gravitational 2BP, total energy level, is introduced to study the 3-body 1-D Coulomb formation. Note that in the gravitational 2BP, the hyperbola is a non-retrievable trajectory type, and it corresponds to an energy level that is greater than zero. By assuming that the charges of the spacecraft are constant, the total energy (kinetic energy and potential energy) of the 3 -body system is constant. The unarrestable motion corresponds to a positive energy level, and the stabilizable motion has a total energy that is negative.

The general relative kinetic energy $T_{\text {rel }}$ is given by Eq. (6.16). Using the symmetric conditions provided above, $T_{\text {rel }}$ is simplified to be

$$
\begin{equation*}
T_{\mathrm{rel}}=\frac{m^{2}}{2 M} \delta \dot{x}_{12}^{2}+\frac{m^{2}}{2 M}\left(\delta \dot{x}_{12}+\delta \dot{x}_{23}\right)^{2}+\frac{m^{2}}{2 M} \delta \dot{x}_{23}^{2}=\frac{M}{3} \delta \dot{x}_{12}^{2} \tag{6.56}
\end{equation*}
$$

where $M=3 m$ is the total formation mass. The electrostatic potential energy of the formation is

$$
\begin{equation*}
V_{\mathrm{e}}=k_{\mathrm{c}} \frac{Q_{12}}{\delta x_{12}}+k_{\mathrm{c}} \frac{Q_{23}}{\delta x_{23}}+k_{\mathrm{c}} \frac{Q_{13}}{\delta x_{12}+\delta x_{23}} \tag{6.57}
\end{equation*}
$$

Utilizing the symmetric motion condition in Eq. (6.54) and Eq. (6.55), $V_{\mathrm{e}}$ is simplified to be

$$
\begin{equation*}
V_{\mathrm{e}}=k_{\mathrm{c}}\left(-\frac{q_{\max }^{2}}{\delta x_{12}}-\frac{q_{\max }^{2}}{\delta x_{12}}+\frac{q_{\max }^{2}}{2 \delta x_{12}}\right)=-\frac{3 k_{\mathrm{c}} q_{\max }^{2}}{2 \delta x_{12}} \tag{6.58}
\end{equation*}
$$

Thus the total energy is obtained by adding up the kinetic energy and potential energy:

$$
\begin{equation*}
E_{\mathrm{t}}=T_{\mathrm{rel}}+V_{\mathrm{e}}=\frac{M}{3} \delta \dot{x}_{12}^{2}-\frac{3 k_{\mathrm{c}} q_{\mathrm{max}}^{2}}{2 \delta x_{12}} \tag{6.59}
\end{equation*}
$$

which has a very simple form due to the symmetric relative motion assumption. Because the charges of the spacecraft are constants in this saturated control discussion, the total energy is also constant. For a stabilizable motion, the total energy $E_{\mathrm{t}}$ should be negative, that is

$$
\begin{equation*}
E_{\mathrm{t}}=\frac{M}{3} \delta \dot{x}_{12}^{2}-\frac{3 k_{\mathrm{c}} q_{\mathrm{max}}^{2}}{2 \delta x_{12}}<0 \tag{6.60}
\end{equation*}
$$

If $E_{\mathrm{t}}<0$, then it is impossible for $\delta x_{12} \rightarrow \infty$. However, if $E_{\mathrm{t}}>0$, then $\delta \dot{x}_{12}$ will approach a positive value as $\delta x_{12} \rightarrow \infty$. Transforming Eq. (6.60) such that only $\delta x_{12}$ and $\delta \dot{x}_{12}$ remain on the left hand side yields the condition

$$
\begin{equation*}
\delta \dot{x}_{12}^{2} \delta x_{12}<\frac{9 k_{\mathrm{c}} q_{\mathrm{max}}^{2}}{2 M}=\frac{3 k_{\mathrm{c}} q_{\mathrm{max}}^{2}}{2 m} \tag{6.61}
\end{equation*}
$$

Eq. (6.61) provides an analytical criterion for the initial states $\delta x_{12}$ and $\delta \dot{x}_{12}$ to result in a stabilizable symmetric motion. From this criterion it can be seen that when the charges and masses of the three spacecraft are set, both the distance and distance rate should be within a certain range to ensure that the symmetric relative motion can be stopped. A bigger charge limit results in a bigger value in the right hand side of the inequality in Eq. (6.61). Thus the area in the $\delta x_{12}-\delta \dot{x}_{12}$ plane that satisfies the criterion is bigger. Note that this criterion is valid only for the symmetric relative motion of the 1-D Coulomb formation. The following discussion will investigate the convergence area of general motions of the 1-D Coulomb formation.

### 6.4.2 Convergence Area For General Cases

The previous subsection derives the converge criterion for the symmetric relative motion by investigating the total energy of the system. Due to the changing polarity of the spacecraft charges, the energy of the system is not constant even though the
magnitude of the charges remain the same. It's very difficult to apply the similar approach as in the symmetric motion to analyze the general convergence area of the saturated control.

Though an analytical solution is difficult to achieve, numerical results are always obtainable. The convergence area can be illustrated by marking each set of initial conditions with which the distance rates converge to zeros in the numerical simulation. Without the assumption of symmetric motion, the initial conditions of the motion contain four independant variables: $\left[\delta x_{12}, \delta x_{23}, \delta \dot{x}_{12}, \delta \dot{x}_{23}\right]$. Thus the convergence area should be configured as a four dimensional region. To illustrate the convergences areas in 2-dimensional plots, the distances and distance rates are illustrated separately. After a certain set of initial $\left[\delta \dot{x}_{12}, \delta \dot{x}_{23}\right]$ is prescribed, the resulting initial $\left[\delta x_{12}, \delta x_{23}\right]$ conditions' area of convergence is illustrated in a 2-D phase plane. And the convergence area of the variables $\left[\delta \dot{x}_{12}, \delta \dot{x}_{23}\right]$ is demonstrated in the similar way in the $\delta \dot{x}_{12}-\delta \dot{x}_{23}$ plane.


Figure 6.4: Area of convergence of $\left(\delta x_{12}, \delta x_{23}\right)$.

Taking the 1-D non-conducting hover track vehicles as an example, let the masses be $m_{1}=m_{2}=m_{3}=10 \mathrm{~kg}$, and the charge limits be $q_{1 \max }=q_{2 \max }=q_{3 \max }=q_{\max }=$ $5 \times 10^{-5} \mathrm{C}$. Let the control parameters be $p_{1}=p_{2}=1 \mathrm{~kg} /\left(\mathrm{C}^{2} \cdot \mathrm{~s}\right)$. Figure 6.4 shows
the convergence areas of the distances $\delta x_{12}, \delta x_{23}$ under different initial distance rates. Figure 6.4(a) shows the case when the initial distance rates $\left[\delta \dot{x}_{12}, \delta \dot{x}_{23}\right]=[0.1,0.1] \mathrm{m} / \mathrm{s}$. The shaded area represents the initial conditions which lead to converged states. It can be seen that the convergence area is not quite symmetric in $\delta x_{12}$ and $\delta x_{23}$ directions. This is because the charge implementation strategy by varying the matrix $[P]$ does not result in symmetric solutions while switching the values of the individual distances $\delta x_{12}$ and $\delta x_{23}$. Figure 6.4(b) shows the convergence area of the $\delta x_{12}-\delta x_{23}$ plane when the initial distance rates are set to be $\left[\delta \dot{x}_{12}, \delta \dot{x}_{23}\right]=[0.1,0.2] \mathrm{m} / \mathrm{s}$. The convergence area shrinks greatly in $\delta x_{23}$ direction. This is because the departing speed $\delta \dot{x}_{23}$ is larger than $\delta \dot{x}_{12}$; it makes $\delta \dot{x}_{23}$ converge much more difficult than $\delta \dot{x}_{12}$.


Figure 6.5: Area of convergence of $\left(\delta \dot{x}_{12}, \delta \dot{x}_{23}\right)$.

Figure 6.5 illustrates two convergence areas of the distance rates in the $\delta \dot{x}_{12}-\delta \dot{x}_{23}$ plane. It can be seen that the convergence area reduces in the direction where the distance increases. The scales of the axes $\delta \dot{x}_{i j}$ range within $[-0.2,0.8] \mathrm{m} / \mathrm{s}$ in the plots. The negative distance rate means the two spacecraft are approaching each other. If the magnitude of the negative distance rate is too big, then the spacecraft are getting close too fast, this may result in collision of spacecraft which is not contained in the scope
of this chapter. Chapter 3 develops the analytical criteria for two spacecraft which are approaching each other to be able to avoid a collision.

### 6.5 Numerical Simulation

A two-stage control strategy has been developed to control the shape of the 1D constrained Coulomb structure. At first the saturated control is used to arrest the relative motion of the spacecraft. After the relative motion has been stabilized, the formation shape controller is employed to make the formation construct a certain shape that is defined by the given desired distances $\left[\delta x_{12}^{*}, \delta x_{23}^{*}\right]$. This section presents some numerical simulation results to show the performance of the control strategy.

The physical parameters of the model are set to be the parameters of a proposed 1-D hover track test bed and are used to test the control algorithm of the 1-D Coulomb structure stabilization control. The masses of the three spacecraft are $m_{1}=m_{2}=m_{3}=$ 10 kg , while the desired shape is given as $\left[\delta x_{12}^{*}, \delta x_{23}^{*}\right]=[4,4] \mathrm{m}$. The separation distances between craft are within 5 meters. Without loss of generality, let the magnitudes of the charges of the spacecraft share a common limit $q_{\max }=5 \times 10^{-5} \mathrm{C}$. Let us choose the initial positions and velocities to be:

$$
\begin{align*}
& {\left[x_{1}, x_{2}, x_{3}\right]=[-3,0,2] \mathrm{m}}  \tag{6.62}\\
& {\left[\dot{x}_{1}, \dot{x}_{2}, \dot{x}_{3}\right]=[-0.04,0,0.04] \mathrm{m} / \mathrm{s}} \tag{6.63}
\end{align*}
$$

Figure 6.6 shows the first stage of the control which arrests the relative motion. The two simulation stage results are illustrated separately because the saturated control has a stronger control forces and the relative motion converges much faster than the time needed in the continuous shape control. The parameters of the saturated regulator are $p_{1}=p_{2}=1 \mathrm{~kg} /\left(\mathrm{C}^{2} \cdot \mathrm{~s}\right)$. The relative distance rates converge to zero in a very short time, and the control charges are always saturated until the distance rates converge. The stability of the control is guaranteed, and if the initial conditions are within the


Figure 6.6: Numerical simulation results of stage I: relative motion regulation.
convergence area presented in the last section, then the relative rates will converge to zero.

Figure 6.7 illustrates the simulation results of the second stage, continuous formation shape control. The parameters of this control are

$$
[K]=\left[\begin{array}{cc}
3.6 & 0  \tag{6.64}\\
0 & 1.8
\end{array}\right] \mathrm{kgm} / \mathrm{s}^{2},[P]=\left[\begin{array}{cc}
14.4 & 0 \\
0 & 7.2
\end{array}\right] \mathrm{kgm} / \mathrm{s}
$$

The values of matrices $[K]$ and $[P]$ are chosen to balance between the overshooting and the response speed. Figure $6.7(\mathrm{a})$ and (b) show the process of the Coulomb structure to converge to the desired shape. Figure 6.7(c) and (d) are the charge histories under different conditions. The chattering issue of the charges is nontrivial in the control process. As mentioned before in the formation shape control section, the chattering effect is partly due to the switching between two possible values of the variable $\tau$. The parameter $\alpha \leq 1$ has been introduced to buffer the switching. With $\alpha=1$, no buffer is acting on the system. When $0<\alpha<1$, the buffer is taking effect. Comparing Figure 6.7(c) and (d), it can be seen that when $\alpha=0.7$, the chattering effect is reduced to some extent. Though the buffer can not totally eliminate the chattering, the benefit is that this approach does not influence the dynamics of the system. This is because


Figure 6.7: Numerical simulation results of stage II: formation shape control.
any value of the variable $\tau$ results in a vector that is within the null space of the input matrix of the control.

### 6.6 Conclusion

A two-stage stable charge feedback control strategy is developed to shape the configuration of the 1-D restricted Coulomb structure. The first stage intends to arrest the relative motion of the formation. A globally stable, but not asymptotically stable saturated control is designed using Lyapunov's direct method. Varying the value of a positive definite matrix used in designing the Lyapunov function rate guarantees real charge solutions. The analytical criterion for a stabilizable symmetric motion is
obtained by evaluating the total energy level of the system. For general cases, the convergence areas of the initial states for stabilizable motions are illustrated numerically. The second stage is a continuous formation shape control. It is used to control the shape of the Coulomb structure to a certain desired configuration. The control is also designed using Lyapunov's method. A minimum charge search routine in the null space of the input matrix is used to solve the control charge implementability problem. The search routine not only makes the charge control law physically implementable, but also results in minimum control charges at every instance. Numerical simulations verify the effectiveness of the control strategy.

This chapter presents two strategies to solve the charge implementation issue: utilizing the null space of the control input matrix (matrix $[C]$ in this chapter); varying the Lyapunov rate matrix (matrix $[P]$ in this chapter). These approaches can be utilized in the further studies of the multiple spacecraft Coulomb virtual structure control problem.

## CHAPTER 7

## THREE-CRAFT NON-EQUILIBRIUM COULOMB VIRTUAL STRUCTURE CONTROL

Chapter 5 studies the two-craft Coulomb virtual structure control in 3-D space. Chapter 6 develops a two-stage stable charge feedback control strategy to shape the configuration of a one-dimensionally constrained 3-craft Coulomb structure. Based on the knowledge provided by these two chapters, this chapter investigates the more general problem of Coulomb virtual structure control: three-craft triangular configuration Coulomb structure control. The objective is to find a stable control strategy to make a three-craft formation stabilize to an arbitrary triangular configuration. The work in this chapter has been presented in Reference [32] and has been submitted to AIAA Journal of Guidance, Control and Dynamics for publication.

### 7.1 EOM of the 3D Coulomb Structure System

Figure 7.1 illustrates the 3-D three-craft Coulomb structure scenario. Assuming the spacecraft are flying in free space and no external forces acting on the system, the EOMs of individual spacecraft are:

$$
\begin{align*}
& m_{1} \ddot{\boldsymbol{R}}_{1}=-k_{\mathrm{c}} \frac{q_{1} q_{2}}{L_{12}^{2}} \hat{\boldsymbol{e}}_{12}-k_{\mathrm{c}} \frac{q_{1} q_{3}}{L_{13}^{2}} \hat{\boldsymbol{e}}_{13}  \tag{7.1a}\\
& m_{2} \ddot{\boldsymbol{R}}_{2}=k_{\mathrm{c}} \frac{q_{1} q_{2}}{L_{12}^{2}} \hat{\boldsymbol{e}}_{12}-k_{\mathrm{c}} \frac{q_{2} q_{3}}{L_{23}^{2}} \hat{\boldsymbol{e}}_{23}  \tag{7.1b}\\
& m_{3} \ddot{\boldsymbol{R}}_{3}=k_{\mathrm{c}} \frac{q_{1} q_{3}}{L_{13}^{2}} \hat{\boldsymbol{e}}_{13}+k_{\mathrm{c}} \frac{q_{2} q_{3}}{L_{23}^{2}} \hat{\boldsymbol{e}}_{23} \tag{7.1c}
\end{align*}
$$



Figure 7.1: Three-dimensional three-craft Coulomb structure.
where $L_{i j}$ is the separation distance between spacecraft- $i(\mathrm{SC} i)$ and spacecraft- $j(\mathrm{SC} j)$, $\hat{\boldsymbol{e}}_{i j}$ is the unit vector point from $\mathrm{SC} i$ to $\mathrm{SC} j$.

Note that using only Coulomb forces can only control the relative motion of spacecraft, not the inertial positions, and the purpose of this work is to control the shape of the Coulomb structure. The shape of a 3-body formation can be completely defined by the three separation distances between any two spacecraft. Here the control goal is defined as making the three separation distances $\left(L_{12}, L_{23}, L_{13}\right)^{T}$ converge to the desired distances $\left(L_{12}^{*}, L_{23}^{*}, L_{13}^{*}\right)^{T}$. The first step is to identify the separation distances' EOMs.

For notational convenience, introduce a vector $\boldsymbol{\xi}$ :

$$
\begin{equation*}
\boldsymbol{\xi}=\left(\xi_{1}, \xi_{2}, \xi_{3}\right)^{T}=k_{\mathrm{c}}\left(\frac{q_{1} q_{2}}{L_{12}^{2}}, \quad \frac{q_{2} q_{3}}{L_{23}^{2}}, \quad \frac{q_{1} q_{3}}{L_{13}^{2}}\right)^{T} \tag{7.2}
\end{equation*}
$$

From the EOMs in Eq. (7.1), the relative positions' EOMs are found to be:

$$
\begin{align*}
\ddot{\boldsymbol{r}}_{12} & =\xi_{1}\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right) \hat{\boldsymbol{e}}_{12}-\xi_{2} \frac{1}{m_{2}} \hat{\boldsymbol{e}}_{23}+\xi_{3} \frac{1}{m_{1}} \hat{\boldsymbol{e}}_{13}  \tag{7.3a}\\
\ddot{\boldsymbol{r}}_{23} & =-\xi_{1} \frac{1}{m_{2}} \hat{\boldsymbol{e}}_{12}+\xi_{2}\left(\frac{1}{m_{2}}+\frac{1}{m_{3}}\right) \hat{\boldsymbol{e}}_{23}+\xi_{3} \frac{1}{m_{3}} \hat{\boldsymbol{e}}_{13}  \tag{7.3b}\\
\ddot{\boldsymbol{r}}_{13} & =\xi_{1} \frac{1}{m_{1}} \hat{\boldsymbol{e}}_{12}+\xi_{2} \frac{1}{m_{3}} \hat{\boldsymbol{e}}_{23}+\xi_{3}\left(\frac{1}{m_{1}}+\frac{1}{m_{3}}\right) \hat{\boldsymbol{e}}_{13} \tag{7.3c}
\end{align*}
$$

Using the facts that $\boldsymbol{r}_{i j}=\boldsymbol{r}_{i}-\boldsymbol{r}_{j}$ and $L_{i j}=\boldsymbol{r}_{i j} \cdot \hat{\boldsymbol{e}}_{i j}$, the separation distances' EOMs are found:

$$
\begin{align*}
& \ddot{L}_{12}=\xi_{1}\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right)+\xi_{2} \frac{\cos \alpha_{2}}{m_{2}}+\xi_{3} \frac{\cos \alpha_{1}}{m_{1}}+\frac{\left\|\dot{\boldsymbol{r}}_{12}\right\|^{2}}{L_{12}}\left(1-\cos ^{2} \angle\left(\boldsymbol{r}_{12}, \dot{\boldsymbol{r}}_{12}\right)\right)  \tag{7.4a}\\
& \ddot{L}_{23}=\xi_{1} \frac{\cos \alpha_{2}}{m_{2}}+\xi_{2}\left(\frac{1}{m_{2}}+\frac{1}{m_{3}}\right)+\xi_{3} \frac{\cos \alpha_{3}}{m_{3}}+\frac{\left\|\dot{\boldsymbol{r}}_{23}\right\|^{2}}{L_{23}}\left(1-\cos ^{2} \angle\left(\boldsymbol{r}_{23}, \dot{\boldsymbol{r}}_{23}\right)\right)  \tag{7.4b}\\
& \ddot{L}_{13}=\xi_{1} \frac{\cos \alpha_{1}}{m_{1}}+\xi_{2} \frac{\cos \alpha_{3}}{m_{3}}+\xi_{3}\left(\frac{1}{m_{1}}+\frac{1}{m_{3}}\right)+\frac{\left\|\dot{\boldsymbol{r}}_{13}\right\|^{2}}{L_{13}}\left(1-\cos ^{2} \angle\left(\boldsymbol{r}_{13}, \dot{\boldsymbol{r}}_{13}\right)\right) \tag{7.4c}
\end{align*}
$$

Starting from the separation distance EOMs in Eq. (7.4), a control strategy is expected to drive the separation distances to desired values, using $\boldsymbol{\xi}$ as the control vector.

### 7.2 Virtual Structure Control Strategy

The goal of the virtual structure control is to make the separation distances converge to the given desired distances:

$$
\begin{equation*}
\left(L_{12}, L_{23}, L_{13}\right)^{T} \rightarrow\left(L_{12}^{*}, L_{23}^{*}, L_{13}^{*}\right)^{T} . \tag{7.5}
\end{equation*}
$$

It is assumed that the desired shape of the 3 -body system is stationary, which indicates that the nominal separation distances $L_{i j}^{*}$ are constant. Using the state vector $\boldsymbol{X}=$ $\left(L_{12}, L_{23}, L_{13}\right)^{T}$, the separation distances' EOMs in Eq. (7.4) are rewritten into a
concise form as:

$$
\ddot{\boldsymbol{X}}=\underbrace{\left[\begin{array}{ccc}
\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right) & \frac{\cos \alpha_{2}}{m_{2}} & \frac{\cos \alpha_{1}}{m_{1}}  \tag{7.6}\\
\frac{\cos \alpha_{2}}{m_{2}} & \left(\frac{1}{m_{2}}+\frac{1}{m_{3}}\right) & \frac{\cos \alpha_{3}}{m_{3}} \\
\frac{\cos \alpha_{1}}{m_{1}} & \frac{\cos \alpha_{3}}{m_{3}} & \left(\frac{1}{m_{1}}+\frac{1}{m_{3}}\right)
\end{array}\right]}_{[B]}\left(\begin{array}{l}
\xi_{1} \\
\xi_{2} \\
\xi_{3}
\end{array}\right)+\underbrace{\left(\begin{array}{c}
\frac{1}{L_{12}}\left\|\dot{\boldsymbol{r}}_{12}\right\|^{2}\left(1-\cos ^{2} \angle\left(\boldsymbol{r}_{12}, \dot{\boldsymbol{r}}_{12}\right)\right) \\
\frac{1}{L_{23}}\left\|\dot{\boldsymbol{r}}_{23}\right\|^{2}\left(1-\cos ^{2} \angle\left(\boldsymbol{r}_{23}, \dot{\boldsymbol{r}}_{23}\right)\right) \\
\frac{1}{L_{13}}\left\|\dot{\boldsymbol{r}}_{13}\right\|^{2}\left(1-\cos ^{2} \angle\left(\boldsymbol{r}_{13}, \dot{\boldsymbol{r}}_{13}\right)\right)
\end{array}\right)}_{\boldsymbol{f}}
$$

The objective of the 3 -craft Coulomb structure shape control is to drive the distances between any two craft to desired values thus to construct a certain triangular shape, i.e. drive $\boldsymbol{X}$ to desired value $\boldsymbol{X}^{*}$, assuming that $\boldsymbol{X}^{*}$ is constant. Define state tracking error vector $\Delta \boldsymbol{X}=\boldsymbol{X}-\boldsymbol{X}^{*}$.

### 7.2.1 3-Side Control Law

Define Lyapunov function candidate as

$$
\begin{equation*}
V=\frac{1}{2} \Delta \boldsymbol{X}^{T}[K] \Delta \boldsymbol{X}+\frac{1}{2} \Delta \dot{\boldsymbol{X}}^{T} \Delta \dot{\boldsymbol{X}} \tag{7.7}
\end{equation*}
$$

where $[K]$ is a $3 \times 3$ positive definite matrix. The derivative of $V$ is

$$
\begin{align*}
\dot{V} & =\Delta \dot{\boldsymbol{X}}^{T}([K] \Delta \boldsymbol{X}+\Delta \ddot{\boldsymbol{X}})  \tag{7.8}\\
& =\Delta \dot{\boldsymbol{X}}^{T}([K] \Delta \boldsymbol{X}+[B] \boldsymbol{\xi}+\boldsymbol{f}) \tag{7.9}
\end{align*}
$$

Prescribe $\dot{V}$ as the following negative semi-definite function:

$$
\begin{equation*}
\dot{V}=-\Delta \dot{\boldsymbol{X}}^{T}[P] \Delta \dot{\boldsymbol{X}} \tag{7.10}
\end{equation*}
$$

Because $[B]$ is a nonsingular matrix, the unique solution to Eq. (7.10) is

$$
\begin{equation*}
\boldsymbol{\xi}=[B]^{-1}(-[K] \Delta \boldsymbol{X}-[P] \Delta \dot{\boldsymbol{X}}-\boldsymbol{f}) \tag{7.11}
\end{equation*}
$$

Charges can be deduced from the definition of $\boldsymbol{\xi}$ in Eq. (7.2) as

$$
\left\{\begin{array}{l}
q_{1}=\sqrt{\frac{a c}{b k_{c}} \frac{\left|L_{12} L_{13}\right|}{\left|L_{23}\right|}}  \tag{7.12}\\
q_{2}=\operatorname{sign}(b c) \sqrt{\frac{a b}{c k_{\mathrm{c}}} \frac{\left|L_{12} L_{23}\right|}{\left|L_{133}\right|}} \\
q_{3}=\operatorname{sign}(c) \sqrt{\frac{b c}{a k_{\mathrm{c}}}} \frac{\left|L_{23} L_{13}\right|}{\left|L_{12}\right|}
\end{array}\right.
$$

Notice that $a \cdot b \cdot c \sim\left(q_{1} q_{2} q_{3}\right)^{2}$, so for implementing this control law with nonimaginary charges, $\boldsymbol{\xi}$ must satisfy

$$
\begin{equation*}
a \cdot b \cdot c \geq 0 \tag{7.13}
\end{equation*}
$$

When $\boldsymbol{\xi}$ does not satisfy the inequality in Eq. (7.13), the control becomes unimplementable. Unfortunately, because $\boldsymbol{\xi}$ is the only solution to Eq. (7.10), nothing more can be done to deal with this implementable problem based on the Lyapunov function given by Eq. (7.7). Thus, the following section seek an alternate control strategy.

### 7.2.2 2-Side Control Strategy

The previous section develops a Lyapunov-based control law that controls the three triangle side-lengths at once. The control is asymptotically stable, but it's not always physically implementable because at times it requires imaginary charges. If we control two sides at once instead of controlling three sides, correspondingly a subset of the state-space EOMs in Eq. (7.6) are considered, then the control input matrix $[B]$ becomes a $2 \times 3$ matrix. Utilizing the null space of the control input matrix, there is a family of solutions that have the same response. An implementable solution can always be found from this solution family. The use of the null space of the input matrix to determine implementable real charge solutions is discussed in Chapter 6.

This section proposes a strategy that always controls the "worst" two sides of the triangle. By continuously switching to control the "worst" two sides, it's expected that the system is stabilized and the state tracking error converge to zero. However, the actual switching strategy must be carefully chosen to avoid making the system unstable.

Define the switching criterion by investigating the three sub-Lyapunov functions as:

$$
\begin{align*}
& V_{a}=\frac{1}{2} k\left(\Delta X_{1}^{2}+\Delta X_{3}^{2}\right)+\frac{1}{2}\left(\Delta X_{1}^{2}+\Delta X_{3}^{2}\right) \triangleq \frac{k}{2} \Delta \boldsymbol{X}_{a}^{T} \Delta \boldsymbol{X}_{a}+\frac{1}{2} \Delta \dot{\boldsymbol{X}}_{a}^{T} \Delta \dot{\boldsymbol{X}}_{a},  \tag{7.14a}\\
& V_{b}=\frac{1}{2} k\left(\Delta X_{1}^{2}+\Delta X_{2}^{2}\right)+\frac{1}{2}\left(\Delta X_{1}^{2}+\Delta X_{2}^{2}\right) \triangleq \frac{k}{2} \Delta \boldsymbol{X}_{b}^{T} \Delta \boldsymbol{X}_{b}+\frac{1}{2} \Delta \dot{\boldsymbol{X}}_{b}^{T} \Delta \dot{\boldsymbol{X}}_{b},  \tag{7.14b}\\
& V_{c}=\frac{1}{2} k\left(\Delta X_{2}^{2}+\Delta X_{3}^{2}\right)+\frac{1}{2}\left(\Delta X_{2}^{2}+\Delta X_{3}^{2}\right) \triangleq \frac{k}{2} \Delta \boldsymbol{X}_{c}^{T} \Delta \boldsymbol{X}_{c}+\frac{1}{2} \Delta \dot{\boldsymbol{X}}_{c}^{T} \Delta \dot{\boldsymbol{X}}_{c} . \tag{7.14c}
\end{align*}
$$

The subscripts ( $a, b, c$ ) denote the errors of the two sides cornered at the (1st, 2nd, 3rd) spacecraft respectively. The final Lyapunov function candidate being activated is chosen to be the largest sub-Lyapunov function:

$$
\begin{equation*}
V_{\mathrm{ctrl}}=\max \left\{V_{a}, V_{b}, V_{c}\right\} . \tag{7.15}
\end{equation*}
$$

Once the control $\boldsymbol{\xi}$ is determined, the motions of the three sides are determined by Eq. (7.6). In order to develop a control algorithm to only stabilize two sides at once, the dynamics of the two sides being controlled are:

$$
\begin{equation*}
\ddot{\boldsymbol{X}}_{\mathrm{ctrl}}=\left[B_{\mathrm{ctrl}}\right] \boldsymbol{\xi}+\boldsymbol{f}_{\mathrm{ctrl}}, \tag{7.16}
\end{equation*}
$$

where $\left[B_{\mathrm{ctrl}}\right]$ is a $2 \times 3$ matrix with the two rows selected from the matrix $[B]$ according to the two sides being controlled.

Taking a first-order time derivative of $V_{\text {ctrl }}$, yields:

$$
\begin{equation*}
\dot{V}_{\mathrm{ctrl}}=\Delta \dot{\boldsymbol{X}}_{\mathrm{ctrl}}^{T}\left(k \Delta \boldsymbol{X}_{\mathrm{ctrl}}+\left[B_{\mathrm{ctrl}}\right] \boldsymbol{\xi}+\boldsymbol{f}_{\mathrm{ctrl}}\right) \tag{7.17}
\end{equation*}
$$

Let $\dot{V}_{\text {ctrl }}$ be the semi-definite function

$$
\begin{equation*}
\dot{V}_{\mathrm{ctrl}}=-\Delta \dot{\boldsymbol{X}}_{\mathrm{ctrl}}^{T}\left[P_{2}\right] \Delta \dot{\boldsymbol{X}}_{\mathrm{ctrl}} \tag{7.18}
\end{equation*}
$$

where $\left[P_{2}\right]$ is a $2 \times 2$ positive definite matrix. Substituting Eq. (7.17) into Eq. (7.18), yields:

$$
\begin{equation*}
\left[B_{\mathrm{ctrl}}\right] \boldsymbol{\xi}=-k \Delta \boldsymbol{X}_{\mathrm{ctrl}}-\boldsymbol{f}_{\mathrm{ctrl}}-\left[P_{2}\right] \Delta \dot{\boldsymbol{X}}_{\mathrm{ctrl}} . \tag{7.19}
\end{equation*}
$$

Note that $\left[B_{\mathrm{ctrl}}\right]$ is a $2 \times 3$ matrix. As mentioned in the beginning of this section, there is a family of solutions of $\boldsymbol{\xi}$ that satisfy the control condition in Eq. (7.19). Let us begin with the minimum norm solution to Eq. (7.19):

$$
\begin{equation*}
\hat{\boldsymbol{\xi}}=\left[B_{\mathrm{ctrl}}\right]^{\dagger}\left(-k \Delta \boldsymbol{X}_{\mathrm{ctrl}}-\boldsymbol{f}_{\mathrm{ctrl}}-\left[P_{2}\right] \Delta \boldsymbol{X}_{\mathrm{ctrl}}\right), \tag{7.20}
\end{equation*}
$$

where $\left[B_{\mathrm{ctrl}}\right]^{\dagger}=\left[B_{\mathrm{ctrl}}\right]^{T}\left(\left[B_{\mathrm{ctr}}\right]\left[B_{\mathrm{ctrl}}\right]^{T}\right)^{-1}$ is the pseudo-inverse of the matrix $\left[B_{\mathrm{ctrl}}\right]$. Note that $\hat{\boldsymbol{\xi}}$ in Eq. (7.20) is the minimum solution to Eq. (7.19) which minimizes the norm of the $\boldsymbol{\xi}$ vector. The general solution to Eq. (7.19) can be written as:

$$
\begin{equation*}
\boldsymbol{\xi}=\hat{\boldsymbol{\xi}}+\gamma \cdot \boldsymbol{b}_{\mathrm{ctrl}}, \tag{7.21}
\end{equation*}
$$

where $\boldsymbol{b}_{\text {ctrl }}$ is a $3 \times 1$ base vector of the null space of the matrix $\left[B_{\text {ctrr }}\right]$. Because $\left[B_{\text {ctrr }}\right]$ is a $2 \times 3$ matrix, it always has a non-empty null space. The scalar parameter $\gamma \in \mathcal{R}$ can be any real number. The flexibility of the value of $\gamma$ provides a single degree of freedom (DOF) that can be utilized to find an implementable (real spacecraft charge) control solution.

With the implementation problem having been narrowed down to finding a proper value of $\gamma$ to make the solution $\boldsymbol{\xi}$ implementable, we rewrite the implementability criterion as:

$$
\begin{equation*}
\xi_{1} \cdot \xi_{2} \cdot \xi_{3} \geq 0 \tag{7.22}
\end{equation*}
$$

Substituting Eq. (7.21) into the criterion, yields:

$$
\begin{equation*}
g(\gamma) \triangleq \xi_{1} \cdot \xi_{2} \cdot \xi_{3}=\left(\hat{\xi}_{1}+\gamma b_{\mathrm{ctrl}}(1)\right)\left(\hat{\xi}_{2}+\gamma b_{\mathrm{ctrl}}(2)\right)\left(\hat{\xi}_{3}+\gamma b_{\mathrm{ctrl}}(3)\right) \geq 0, \tag{7.23}
\end{equation*}
$$

where $\hat{\xi}_{i}$ is given by the minimum norm solution in Eq. (7.20). The next step is to find a value of $\gamma$ that satisfies the inequality $g(\gamma) \geq 0$. Note that $g(\gamma)$ is a cubic equation of $\gamma$. The two typical cases of the function $g(\gamma)$ are illustrated in Figure 7.2. In both cases, there are two continuous intervals of $\gamma$ that make $g(\gamma) \geq 0$. This indicates that there always exists a family of solutions that make the 2-side control implementable with the same dynamical behavior.


Figure 7.2: Examples of $g(\gamma)$ function in two cases.

Because there is an infinite number of solutions that make the control implementable, a solution is chosen which minimizes the spacecraft charge magnitudes to simplify the technical implementation of this charge control solution. A charge cost function is defined:

$$
\begin{equation*}
J(\gamma)=\sum_{i=1}^{3} q_{i}^{2} . \tag{7.24}
\end{equation*}
$$

Chapter 6 develops an algorithm based on Newton's method to search the optimal solution of $\gamma$ that minimizes the cost function $J(\gamma)$. The same algorithm is applied here to determine the charge-optimal solution.

For a switched Lyapunov-based control, the stability needs to be reevaluated because the switching may introduce discontinuity to the Lyapunov candidate functions. The following property states that if the control charges can switch infinitely fast then the switched control with the switch strategy given by Eq. (7.15) is stable.

Property 1 The switched control strategy with the switch rule given by Eq. (7.15) is stable if $\dot{V}_{\text {ctrl }}<0$ and the control charges are capable to switch infinitely fast, which indicates the switching happens when

$$
\begin{equation*}
V_{\mathrm{ctrl}}=\max \left\{V_{k} \mid V_{k}=V_{a}, V_{b}, V_{c} \text { and } V_{k} \neq V_{\mathrm{ctrl}}\right\} \tag{7.25}
\end{equation*}
$$

Proof Because the control can switch infinitely fast, the Lyapunov function being controlled $V_{\text {ctrl }}$ is continuous. Note that $\dot{V}_{\text {ctrl }} \leq 0$, the system is stable by Lyapunov stability theorem.

In the ideal case with the control charges can switch infinitely fast, the Lyapunov function being controlled is continuous and non-increasing. However, in practice the control frequency is always limited resulting control cycles of a finite duration. The discrete control time step makes the Lyapunov function being controlled discontinuous at the switch point. This discontinuity breaks down the stability proof based on continuous Lyapunov function. The next section utilizes a multiple Lyapunov function analysis tool to analyze the stability of the switched system and develops a stable switch strategy with present of the limited control time step.

### 7.3 Multiple Lyapunov Functions Analysis

The last section designs a switching control strategy that always controls the "worst" two sides of the triangle, with the "worst" two sides defined by the corresponding Lyapunov function candidates. Stability is ensured if the switching can occur infinitely fast. The action of the switching may cause stability issues if the switching occurs over finite time steps. The tracking error of the uncontrolled triangle side can become the largest error during the finite control interval.

Multiple Lyapunov functions for switched systems is a tool to analyze this type of systems with discretely switched control objectives. [33] Before analyzing the switched system, it's necessary to define several concepts.

Definition A switched system is a simple case of a hybrid system that is of multi-modal, while the system switches in a way that there are finite switches in finite time. [33]

Definition Control cycle period is the time period while the control has to be constant without updating, it's limited by the hardware components such as sensors and actuators. The value of the control cycle period is constant.

Definition Switch cycle period is the time period during when the active Lyapunov function hasn't been switched. The value of the switch cycle period is not constant, the minimum possible value is equal to the control cycle period.

The switched control developed in the last section switches according to the three Lyapunov functions defined in Eq. (7.14). Now the switch frequency is constrained by the control cycle period. The maximum switch frequency is the inverse of the control cycle period. This satisfies the definition of the switched system that there are finite switches in finite time.

### 7.3.1 Stability Analysis

The stability of a switched system can not be characterized using only the Lyapunov stability theorem of a continuous system. Even when all the Lyapunov function rates of the activated models are negative semi-definite, the system can still be unstable due to the control objective switching.

Figure 7.3 shows a simulation example of the three-body Coulomb virtual structure control using the continuous control strategy developed in the previous section, but implemented with finite control cycles. Figure 7.3(a) shows the distance errors, Figures 7.3(b)-7.3(d) show the Lyapunov functions in different time ranges. The plots show that the system is stable during Region 1, but unstable during Region 2. Special tools should be engaged to explain and analyze this behavior.

Branicky's contribution in Reference 33 is a milestone in analyzing nonlinear hybrid system. He proves several theorems that justify the stability of different hybrid systems based on Lyapunov's stability theorem. This chapter employs Theorem 2.3


Figure 7.3: Simulation example of the unstable switch control strategy.
from Reference 33 repeated here for clarity:

Theorem 8 (Theorem 2.3 in Reference 33) Suppose we have candidate Lyapunov functions $V_{i}, i=1, \cdots, N$ and vector fields $\dot{x}=f_{i}(x)$. Let $\mathcal{S}$ be the set of all switching sequences associated with the system.

If for each $S \in \mathcal{S}$ we have that for all $i, V_{i}$ is Lyapunov-like for $f_{i}$ and $x_{S}(\cdot)$ over $S \mid i$, then the system is stable in the sense of Lyapunov.
where "Lyapunov-like function" is defined as:

Definition (Reference 33) Given a strictly increasing sequence of times $T$ in $\mathcal{R}$, we say that $V$ is Lyapunov-like for function $f$ and trajectory $x(\cdot)$ over $T$ if:

- $\dot{V} \leq 0$ when it's activated
- V is monotonically nonincreasing on $\mathcal{E}(T)$
where $\mathcal{E}(T)$ denotes the even sequence of $T: t_{0}, t_{2}, t_{4}, \cdots$.

Theorem 8 explains the behavior in Figure 7.3. Figure 7.3(c) shows a snapshot at Region 1. It can be seen that at every other switching time, each Lyapunov function candidate is less than its value at the time point that is two switching cycles before. By Theorem 8, the Lyapunov function candidates $\left(V_{a}, V_{b}, V_{c}\right)$ are Lyapunov-like and the system is stable in this region. Figure 7.3(d) is a snapshot during Region 2. In this case, the control switches at the maximum frequency and the switching cycle period is equal to the control cycle period. Even though during each control cycle the controlled Lyapunov function is decreasing, the un-controlled Lyapunov functions increase faster than the controlled Lyapunov function's decreasing rate. At every other switching time, each Lyapunov function candidate is greater than its value at the time that is two switching periods earlier. So the Lyapunov function candidates $\left(V_{a}, V_{b}, V_{c}\right)$ is not Lyapunov-like during Region 2, and the system is unstable in this region.

The stability of the switched control strategy given by Eq. (7.15) and Eq. (7.21) is not guaranteed because the Lyapunov function candidates ( $V_{a}, V_{b}, V_{c}$ ) are not guaranteed to be Lyapunov-like.

### 7.3.2 Switched Control Stability Requirements

Theorem 8 explains why the switched control strategy given by Eq. (7.15) and Eq. (7.21) can be unstable at times. This section improves the control strategy to make the Lyapunov function candidates $V_{a}-V_{c}$ satisfy the Lyapunov-like conditions, such that the system is made stable even with discrete non-zero control cycles.

Assume that during one switching cycle $V_{\beta}$ is the controlled Lyapunov function. The corresponding two sides being controlled are denoted as $i^{\text {th }}$ and $j^{\text {th }}$ sides, the
uncontrolled side is the $k^{\text {th }}$ side. Here "uncontrolled" does not mean the control won't affect the $k^{\text {th }}$ side, but the $k^{\text {th }}$ is not taken into consideration in developing the control algorithm. Note that when $V_{\beta}$ is under control, the errors in the $i^{\text {th }}$ and $j^{\text {th }}$ sides are decreasing, but the trend of the error in the $k^{\text {th }}$ side is undetermined.

Figure 7.3(d) shows an example that when $V_{a}$ is decreasing, $V_{b}$ and $V_{c}$ are increasing at a very high rate. This means that the errors in the $L_{12}$ and $L_{13}$ sides are decreasing, but the error in the $L_{23}$ side is increasing dramatically and destroys the stability of the system. To ensure stability of the system, the uncontrolled side's behavior can not be neglected.

Note that the control in Eq. (7.21) makes the errors in both of the two sides $i^{\text {th }}$ and $j^{\text {th }}$ decreasing. The error in the uncontrolled side needs to be investigated. Define three error functions in the same form as the Lyapunov function candidates:

$$
\begin{equation*}
V_{1}=\frac{k}{2} \Delta X_{1}^{2}+\frac{1}{2} \Delta \dot{X}_{1}^{2}, \quad V_{2}=\frac{k}{2} \Delta X_{2}^{2}+\frac{1}{2} \Delta \dot{X}_{2}^{2}, \quad V_{3}=\frac{k}{2} \Delta X_{3}^{2}+\frac{1}{2} \Delta \dot{X}_{3}^{2} . \tag{7.26}
\end{equation*}
$$

Without loss of generality, rearrange the state vector in the form

$$
\begin{equation*}
\boldsymbol{X}=\binom{\boldsymbol{X}_{c t r l}}{X_{u c}} \tag{7.27}
\end{equation*}
$$

where $\boldsymbol{X}_{c t r l}$ is composed of two distance errors corresponding to the two controlled side, $X_{u c}$ denote the distance error of the uncontrolled side. Correspondingly the EOM is rewritten to be:

$$
\binom{\ddot{\boldsymbol{X}}_{c t r l}}{\ddot{X}_{u c}}=\left[\begin{array}{c}
{\left[B_{c t r l}\right]}  \tag{7.28}\\
\boldsymbol{B}_{u c}
\end{array}\right]\left(\hat{\boldsymbol{\xi}}+\gamma \boldsymbol{b}_{c t r l}\right)+\binom{\boldsymbol{f}_{c t r l}}{f_{u c}}
$$

where $\boldsymbol{B}_{u c}$ is a $1 \times 3$ vector that is the line in the matrix $[B]$ corresponding to the uncontrolled side, $f_{u c}$ is the component of the vector $\boldsymbol{f}$ corresponding to the uncontrolled
side. Substituting $\hat{\boldsymbol{\xi}}$ in Eq. (7.20) into Eq. (7.28) and carrying out the algebra, yields:

$$
\begin{equation*}
\binom{\ddot{\boldsymbol{X}}_{c t r l}}{\ddot{X}_{u c}}=\binom{-k \Delta \boldsymbol{X}_{c t r l}-\left[P_{2}\right] \Delta \dot{\boldsymbol{X}}_{c t r l}}{\boldsymbol{B}_{u c}\left[B_{c t r l}\right]^{\dagger}\left(-k \Delta \boldsymbol{X}_{c t r l}-\left[P_{2}\right] \Delta \dot{\boldsymbol{X}}_{c t r l}-\boldsymbol{f}_{c t r l}\right)+\gamma \boldsymbol{B}_{u c} \boldsymbol{b}_{c t r l}+f_{u c}} \tag{7.29}
\end{equation*}
$$

Taking a time derivative of the error function of the uncontrolled side $V_{u c}$ and substituting $\ddot{X}_{u c}$, yield:

$$
\begin{align*}
\dot{V}_{u c} & =k \Delta \dot{X}_{u c}\left(\Delta X_{u c}+\Delta \ddot{X}_{u c}\right) \\
& =k \Delta \dot{X}_{u c}\left(\Delta X_{u c}+\boldsymbol{B}_{u c}\left[B_{c t r l}\right]^{\dagger}\left(-k \Delta \boldsymbol{X}_{c t r l}-\left[P_{2}\right] \Delta \dot{\boldsymbol{X}}_{c t r l}-\boldsymbol{f}_{c t r l}\right)+\gamma \boldsymbol{B}_{u c} \boldsymbol{b}_{c t r l}+f_{u c}\right) . \tag{7.30}
\end{align*}
$$

Eq. (7.30) shows that the sign of the uncontrolled side's error is undetermined. Even though there are two parameters [ $P_{2}$ ] and $\gamma$ that can be adjusted, this flexibility does not guarantee there exists a solution to make $\dot{V}_{u c}$ negative because in some cases controlling three sides is impossible.

To find a way to solve this problem, it is beneficial to take a closer look at the unstable situation shown in Figure $7.3(\mathrm{~d})$. Note that the three Lyapunov function candidates are actually the combinations of the error functions:

$$
\begin{equation*}
V_{a}=V_{1}+V_{3}, \quad V_{b}=V_{1}+V_{2}, \quad V_{c}=V_{2}+V_{3} . \tag{7.31}
\end{equation*}
$$

Figure 7.4 shows the details of the Lyapunov function candidates and the error functions during several unstable switches. In Figure 7.4(a), during the $n^{\text {th }}$ switch cycle, $\dot{V}_{c}<0$ while $\dot{V}_{a}$ and $\dot{V}_{b}$ are positive. $\dot{V}_{c}<0$ indicates $\dot{V}_{2}<0$ and $\dot{V}_{3}<0$. This is verified by Figure $7.4(\mathrm{~b})$. So $V_{a, b}>0$ is due to the excessive increasing of $V_{1}$, as shown in Figure 7.4(b). At the beginning of the next control cycle, $(n+1)^{\text {th }}$ control cycle, it is identified that $V_{a}$ is the largest Lypunov function candidate. According to the switch strategy in Eq. (7.15), the controller switches to control $V_{a}$ which indicates $V_{1,3}<0$ as shown in Figure 7.4(b). Focusing on $V_{1}$ in Figure 7.4(b), one can see that during the


Figure 7.4: Unstable switch analysis.
$(n+1)^{\text {th }}$ control cycle, $V_{1}$ is controlled such that $\dot{V}_{1}<0$. But the rate of decreasing of $V_{1}$ is smaller than its increasing rate during the $n^{\text {th }}$ control cycle. This results in that at the next switch time (at the point $C$ in Figure 7.4(b)), $V_{1}$ hasn't decreased to the same level as the value at the beginning of the $n^{\text {th }}$ control cycle (at the point $A$ ). That is $V_{1}^{(C)}>V_{1}^{(A)}$. According to Branicky's theorem in Theorem 8, $V_{1}$ is not Lyapaunov-like and the stability is not guaranteed.

By the above analysis, it can be concluded that the instability comes from two sources:
(1) The decreasing rate of the error function of the new controlled side is not big enough to compensate for its increased amount during the last control cycle.
(2) The increasing rate of the error function of the new uncontrolled side is too big.

Upon entering a new control objective switch, both the new uncontrolled and the new controlled sides' error functions need to be taken care of to ensure the Lyapunov function candidates to be Lyapunov-like. Corresponding to Figure 7.4(b), the magnitude of the slope of $V_{1}$ during the $(n+1)^{\text {th }}$ control cycle should be greater than the slope during the $n^{\text {th }}$ control cycle. The increasing rate of $V_{2}$ during the $(m+1)^{\text {th }}$ control cycle should be less than its decreasing rate during the $n^{\text {th }}$ control cycle. Figure 7.5 illustrates this idea. In this way, $V_{1}^{(C)}<V_{1}^{(A)}$ and $V_{2}^{(C)}<V_{2}^{(A)}$. $V_{3}$ is always being controlled during the two control cycles so it's automatically satisfied that $V_{3}^{(c)}<V_{3}^{(A)}$. Thus all of the Lyapunov function candidates are Lyapunov-like during the two control cycles.

To take care of the new controlled side, which indicates this side was uncontrolled in the last control cycle, the first step is to determine the requirement to remain Lyapunov-like for this side. Let $V_{m}$ denote the new controlled side's error function. The requirement for this side to be Lyapunov-like is that the change of the corresponding error function in the new switch cycle $\Delta V_{m}^{(n+1)}$ should be less than its change in the


Figure 7.5: Hand-Drawn Illustration of the new switch strategy effect.
previous switch cycle $\Delta V_{m}^{(n)}$. This can be expressed mathematically in the way:

$$
\begin{equation*}
\int_{(n+1)} \dot{V}_{m}^{(n+1)} d t<-\Delta V_{m}^{(n)} \tag{7.32}
\end{equation*}
$$

where $\int_{(n+1)}$ means the integration across the $(n+1)^{\text {th }}$ switch cycle. Because the control cycle period is very small, the inequality in Eq. (7.32) is approximated by

$$
\begin{equation*}
\dot{V}_{m}^{(n+1)} \Delta t<-\Delta V_{m}^{(n)}, \tag{7.33}
\end{equation*}
$$

where $\Delta t$ is the control cycle period which is constant. This requires the error function rate $\dot{V}_{m}^{(n+1)}$ should be less than a certain value:

$$
\begin{equation*}
\dot{V}_{m}^{(n+1)}<-\Delta V_{m}^{(n)} / \Delta t . \tag{7.34}
\end{equation*}
$$

Because the subscript $m$ denotes the new controlled side, $\dot{V}_{m}^{(n+1)}$ is determined to be negative. If $\Delta V_{m}^{(n)}$ is negative which means $V_{m}$ decreases in the $n^{\text {th }}$ control cycle, then the requirement in Eq. (7.34) is automatically satisfied. Otherwise, a strategy that
makes the inequality in Eq. (7.36) always satisfied is expected. Taking a time derivative of $V_{m}$ then substituting the EOM of $X_{m}$ in Eq. (7.28) yields:

$$
\begin{equation*}
\dot{V}_{m}=\Delta \dot{X}_{m}\left(k \Delta X_{m}+\boldsymbol{B}_{m}\left[B_{c t r l}\right]^{\dagger}\left(-k \Delta \boldsymbol{X}_{c t r l}-\left[P_{2}\right] \Delta \dot{\boldsymbol{X}}_{\text {ctrl }}-\boldsymbol{f}_{\text {ctrl }}\right)+\boldsymbol{f}_{m}\right) \tag{7.35}
\end{equation*}
$$

In this expression of $\dot{V}_{m}$ only the control coefficients $k$ and $\left[P_{2}\right]$ are not dependent on the states and can be utilized to adjust the value of $\dot{V}_{m}$. We choose to change the matrix $\left[P_{2}\right]$ to make the error functions to be Lyapunov-like. Substituting Eq. (7.35) into the inequality in Eq. (7.36), yields:

$$
\begin{align*}
\Delta \dot{X}_{m} \boldsymbol{B}_{m}\left[B_{c t r l}\right]^{\dagger}\left[P_{2}\right] \Delta \dot{\boldsymbol{X}}_{c t r l}>\Delta & \dot{X}_{m}\left(k \Delta X_{m}+f_{m}\right. \\
& \left.+\boldsymbol{B}_{m}\left[B_{c t r l}\right]^{\dagger}\left(-k \Delta \boldsymbol{X}_{c t r l}-\boldsymbol{f}_{c t r l}\right)\right)+\frac{\Delta V_{m}^{(n)}}{\Delta t} . \tag{7.36}
\end{align*}
$$

The inequality in Eq. (7.36) is the requirement for the matrix $\left[P_{2}\right.$ ] that ensures the error function of the new controlled side is Lyapunov-like. The requirement for the new uncontrolled side is similar:

$$
\begin{align*}
& \Delta \dot{X}_{u c} \boldsymbol{B}_{u c}\left[B_{c t r l}\right]^{\dagger}\left[P_{2}\right] \Delta \dot{\boldsymbol{X}}_{c t r l}>\Delta \dot{X}_{u c}\left(k \Delta X_{u c}+f_{u c}\right. \\
&\left.+\boldsymbol{B}_{u c}\left[B_{c t r l}\right]^{\dagger}\left(-k \Delta \boldsymbol{X}_{c t r l}-\boldsymbol{f}_{c t r l}\right)\right)+\frac{\Delta V_{u c}^{(n)}}{\Delta t} . \tag{7.37}
\end{align*}
$$

The inequalities in Eqs. (7.36), (7.37) are two conditions that guarantees the error functions to be Lyapunov-like. Note that the matrix $\left[P_{2}\right]$ should be positive definite, so there are three requirements for $\left[P_{2}\right]$ that ensures a globally stable switched control.

### 7.3.3 Stable Switched Strategy

The previous section determined three requirements that ensured a stable switched control. This section develops a new switched control strategy that implements the stability requirements found in Eqs. (7.36) and (7.37). Above all, the existence of solutions that satisfy the stability requirements needs to be investigated. Let us begin with introducing an asymmetric positive definite matrix.

Property 2 A $2 \times 2$ matrix $[A]$ in the form

$$
[A]=\left[\begin{array}{cc}
A_{11} & A_{12}  \tag{7.38}\\
-A_{12} & A_{22}
\end{array}\right]
$$

is a positive definite matrix if and only if:

$$
\begin{equation*}
A_{11}>0, \quad A_{22}>0 \tag{7.39}
\end{equation*}
$$

Proof The symmetric part of the matrix $[A]$ is:

$$
\left[A_{s}\right]=\frac{1}{2}[A]+\frac{1}{2}[A]^{T}=\left[\begin{array}{cc}
A_{11} & 0  \tag{7.40}\\
0 & A_{22}
\end{array}\right]
$$

It is evident that the symmetric matrix $\left[A_{s}\right]$ is positive definite if and only if $A_{11}>0$ and $A_{22}>0$. A necessary and sufficient condition for a real matrix to be positive definite is that its symmetric part is positive definite. Thus the matrix $[A]$ is positive definite if and only if $A_{11}>0$ and $A_{22}>0$.

This form of a positive definite matrix is more general than symmetric positive definite matrices. This provides more flexibilities in solving the inequalities in Eqs. (7.36) and (7.37). Note that the inequalities in Eqs. (7.36) and (7.37) can be written in the general form:

$$
\begin{equation*}
\boldsymbol{a}^{T}\left[P_{2}\right] \boldsymbol{b}>c, \tag{7.41}
\end{equation*}
$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are two 2-dimensional vectors, $c$ is a real number that equals the right hand side (RHS) of the inequalities. The following theorem studies the existence of the solutions to this inequality.

Theorem 9 Assume a positive definite matrix in the form:

$$
[A]=\left[\begin{array}{cc}
A_{11} & A_{12}  \tag{7.42}\\
-A_{12} & A_{22}
\end{array}\right]
$$

where $A_{11}$ and $A_{22}$ are positive. Define two arrays: $\boldsymbol{a}=\left[a_{1}, a_{2}\right]^{T}$ and $\boldsymbol{b}=\left[b_{1}, b_{2}\right]^{T}$. If the following two inequalities and one equation do not happen at the same time:

$$
\begin{align*}
& a_{1} b_{1}<0,  \tag{7.43a}\\
& a_{2} b_{2}<0  \tag{7.43b}\\
& a_{2} b_{1}=a_{1} b_{2}, \tag{7.43c}
\end{align*}
$$

then for any $c \in \mathbb{R}$, there always exists a solution of the matrix $[A]$ that satisfies the inequality:

$$
\begin{equation*}
\boldsymbol{a}^{T}[A] \boldsymbol{b}>c \tag{7.44}
\end{equation*}
$$

Proof It needs to be proved that the solution of $[A]$ exists under the following two cases:
(1) $a_{2} b_{1} \neq a_{1} b_{2}$,
(2) $a_{2} b_{1}=a_{1} b_{2}$ and $a_{1} b_{1}>0$ and/or $a_{2} b_{2}>0$.

Carrying out the algebra in the inequality in Eq. (7.42), yields:

$$
\begin{equation*}
a_{1} b_{1} A_{11}+\left(a_{1} b_{2}-a_{2} b_{1}\right) A_{12}+a_{2} b_{2} A_{22}>c \tag{7.45}
\end{equation*}
$$

Note that the requirements for $A_{i j}$ are $A_{11}>0$ and $A_{22}>0$, the third element $A_{12}$ can be any real number. Next the existence of the positive definite matrix $[A]$ is proven under the enumerated two cases.

Case 1: $a_{2} b_{1} \neq a_{1} b_{2}$. When $a_{2} b_{1} \neq a_{1} b_{2}$, the element $A_{12}$ can be used to adjust the value of the left hand side (LHS) of the inequality in Eq. (7.45). If $a_{1} b_{2}>a_{2} b_{1}$, then any real value of $A_{12}$ that satisfies:

$$
\begin{equation*}
A_{12}>\frac{c-a_{1} b_{1} A_{11}-a_{2} b_{2} A_{22}}{a_{1} b_{2}-a_{2} b_{1}} \tag{7.46}
\end{equation*}
$$

is a solution to the inequality in Eq. (7.42) while preserving the positive definiteness of the matrix $[A]$. Alternatively if $a_{1} b_{2}<a_{2} b_{1}$, then any real value of $A_{12}$ that satisfies:

$$
\begin{equation*}
A_{12}<\frac{c-a_{1} b_{1} A_{11}-a_{2} b_{2} A_{22}}{a_{1} b_{2}-a_{2} b_{1}} \tag{7.47}
\end{equation*}
$$

is a solution to the inequality in Eq. (7.42).
Case 2: $a_{2} b_{1}=a_{1} b_{2}$ and $a_{1} b_{1}>0$ and/or $a_{2} b_{2}>0$. When $a_{2} b_{1}=a_{1} b_{2}$, the inequality in Eq. (7.45) simplifies to

$$
\begin{equation*}
a_{1} b_{1} A_{11}+a_{2} b_{2} A_{22}>c \tag{7.48}
\end{equation*}
$$

Because either $a_{1} b_{1}>0$ or $a_{2} b_{2}>0$, without loss of generality it's supposed $a_{1} b_{1}>0$. Solving for $A_{11}$ from the inequality in Eq. (7.48), yields:

$$
\begin{equation*}
A_{11}>\frac{1}{a_{1} b_{1}}\left(c-a_{2} b_{2} A_{22}\right) . \tag{7.49}
\end{equation*}
$$

The inequality in Eq. (7.49) does not conflict with the requirement that $A_{11}>0$. Thus any value of $A_{11}$ that satisfies:

$$
\begin{equation*}
A_{11}>\max \left\{\frac{1}{a_{1} b_{1}}\left(c-a_{2} b_{2} A_{22}\right), 0\right\} \tag{7.50}
\end{equation*}
$$

is a solution to the inequality in Eq. (7.42).

Theorem 9 proves the existence of solutions to the inequalities in Eqs. (7.36) and (7.37) unless the condition in Eq. (7.43) occurs. Note that the two inequalities and one equation in Eq. (7.43) are rarely to happen at the same time. Moreover, $a_{1} b_{2}=a_{2} b_{1}$ is a transient state. Generally this situation is neglectable.

Now all steps are in place to lay out the new stable switched charge control strategy. Note that the old switched control strategy works well most of the time, the temporary loss of stability happens due to the discrete control time steps. The first switch strategy given by Eq. (7.15) is still valid unless the Lyapunov-like condition being violated.

Beginning a new control cycle, there are three possible combinations of the controlled sides. One of them corresponds to no switching case, the other two correspond to two switched control cases. For the notational convenience, denote the three possibilities as "no switching", "switch-1" and "switch-2". When an unstable switching,


Figure 7.6: Stable switch strategy flowchart.
which means the switching does not satisfy the requirements in Eqs. (7.36) and (7.37), is detected, it is easier to change the new controlled side than to change the value of the matrix $\left[P_{2}\right]$. Based on this rule, a new switch strategy is developed and shown in Figure 7.6.

Figure 7.6 illustrates the strategy of switching. The details of calculating the value of the matrix $\left[P_{2}\right]$ is not illustrated. Upon changing the value of $\left[P_{2}\right]$, it's better to start with simpler diagonal form. If the diagonal form of $\left[P_{2}\right]$ matrix cannot provide a stable switched control, then the more complex asymmetric matrix form as shown in Eq. (7.42) is sought.

### 7.4 Numerical Simulations

This section presents numerical simulations to show the effectiveness and performance of the stable switched 3 -craft charge control. The desired shape here is triangular configuration. For notational convenience, the old controller with the switching strategy given by Eq. (7.15) is called Controller-1, the new stable controller with the switching
strategy given by Figure 7.6 is called Controller-2.
Both Controller-1 and Controller-2 are used to control the 3-craft Coulomb virtual structure. Under the same initial conditions the performances of the two controllers can be compared. The response of the system is different in different situations. When the initial errors and separation distances are large, the control charges levels are large. The following two simulation cases illustrate the behavior of the system under two controllers. In all of the simulations the masses of the three spacecraft are the same:

$$
\begin{equation*}
m_{1}=m_{2}=m_{3}=50 \mathrm{~kg} . \tag{7.51}
\end{equation*}
$$

### 7.4.1 Big Control Effort Case

The initial positions and velocities of the three spacecraft are

$$
\left\{\begin{array}{l}
\boldsymbol{r}_{1}=[9,-2,0]^{T} \mathrm{~m}  \tag{7.52}\\
\boldsymbol{r}_{2}=[0,-4,0]^{T} \mathrm{~m} \\
\boldsymbol{r}_{3}=[-2,-2,0]^{T} \mathrm{~m}
\end{array}, \quad\left\{\begin{array}{l}
\dot{\boldsymbol{r}}_{1}=[0,0.01,0]^{T} \mathrm{~m} / \mathrm{s} \\
\dot{\boldsymbol{r}}_{2}=[0,0,0]^{T} \mathrm{~m} / \mathrm{s} \\
\dot{\boldsymbol{r}}_{3}=[0,-0.01,0]^{T} \mathrm{~m} / \mathrm{s}
\end{array} .\right.\right.
$$

The expected triangular shape of the virtual structure is defined by the separation distances:

$$
\begin{equation*}
\boldsymbol{X}^{*}=[6,7,5]^{T} \mathrm{~m} . \tag{7.53}
\end{equation*}
$$

The proportional feedback coefficients are:

$$
\begin{equation*}
k=0.0003 \mathrm{~s}^{-2} . \tag{7.54}
\end{equation*}
$$

The nominal value of the matrix $\left[P_{2}\right]$ is

$$
\left[P_{2}^{*}\right]=\left[\begin{array}{cc}
0.02 & 0  \tag{7.55}\\
0 & 0.02
\end{array}\right] \mathrm{s}^{-1} .
$$

Note that the value of the $\left[P_{2}\right]$ matrix is varying using Controller- 2 .
Figure 7.7 shows the responses of the system under the two different control strategy. Comparing the separation distance errors in Figures 7.7(a) and 7.7(b), it is


Figure 7.7: Big control effort simulations.
evident that the stable switched control strategy performs better than the unstable switched control. Using this set of initial states and controller parameters, the old controller assuming continuos switching capabilities cannot stabilize the distance errors to zero, while the stable switched control strategy with finite control cycles stabilizes the errors near zero. Because the rotating triangular configuration is not an equilibrium solutions, the errors cannot converge perfectly to zero. The smaller the control cycles are, the smaller the final state errors will become.

The error functions' histories in Figure 7.7(c) explains the behavior of the continuousswitching controller. During the time around $700-1000$ s, the controller switches at the highest frequency and the error functions do not satisfy the Lyapunov-like conditions. The details are similar to Figure 7.4(b). Thus this is an unstable part of the response. Figure 7.7(d) shows that under the control of the stable switched control strategy, the error functions drop to very low level $\left(10^{-7} \mathrm{~m}^{2} / \mathrm{s}^{2}\right)$ within 1000 s , but won't really decrease to zero. This means the system is stable, but not asymptotically stable as explained above.

Figures $7.7(\mathrm{e})$ and $7.7(\mathrm{f})$ show the charge histories of the two simulations. It can be seen that under the control of the stable switched controller, the charge histories have more spikes than that of the unstable switched control. This is due to the variation of the matrix $\left[P_{2}\right]$ in the stable switched control. Despite the spikes in the charge histories, it can be seen that after the distance errors settle down (after 800s as shown in Figure 7.7(b)), the control charge level that holds the spinning triangle is around $5 \mu \mathrm{C}$. But at the beginning the charge level goes up to $90 \mu \mathrm{C}$ which is not practically implementable. The practically implementable charge level is within $10 \mu \mathrm{C}$. This simulation case is aggressive. The intention is to show the different behaviors under different situations.

### 7.4.2 Small Control Effort Case

In this simulation, the initial errors and the separation distances are small thus the controllers require small charge levels. The initial conditions are set as

$$
\left\{\begin{array}{l}
\boldsymbol{r}_{1}=[2,0,0] \mathrm{m}  \tag{7.56}\\
\boldsymbol{r}_{2}=[0,-4,0] \mathrm{m} \\
\boldsymbol{r}_{3}=\left[\begin{array}{ll}
-2,-2,0] \mathrm{m}
\end{array} \quad, \quad\left\{\begin{array}{l}
\dot{\boldsymbol{r}}_{1}=[0,0.002,0] \mathrm{m} / \mathrm{s} \\
\dot{\boldsymbol{r}}_{2}=[0,0,0] \mathrm{m} / \mathrm{s} \\
\dot{\boldsymbol{r}}_{3}=\left[\begin{array}{ll}
0, & -0.002,0] \mathrm{m} / \mathrm{s}
\end{array}\right.
\end{array} . . \begin{array}{l}
0,
\end{array}\right] .\right.
\end{array}\right.
$$

The expected separation distances are given by:

$$
\begin{equation*}
\boldsymbol{X}^{*}=[4,, 4,4]^{T} \mathrm{~m} \tag{7.57}
\end{equation*}
$$

The controller coefficients are

$$
\begin{equation*}
k=0.0003 \mathrm{~s}^{-2}, \quad\left[P_{2}^{*}\right]=\operatorname{diag}(0.005,0.005) \mathrm{s}^{-1} \tag{7.58}
\end{equation*}
$$

Figure 7.8 shows the simulation results under the two controllers. Figures 7.8(a), $7.8(\mathrm{c})$ and $7.8(\mathrm{e})$ show the results of the simulation using Controller-1. The distance error history in Figure 7.8 (a) shows that at the beginning 2000s, the errors are staying at high level. The error functions shown in Figure $7.8(\mathrm{e})$ verify that during $[0,2000]$ s, there are several temporary unstable regions where both of the three error functions are increasing. After 2000s, the distance errors are decreasing significantly.

Figures $7.8(\mathrm{~b}), 7.8(\mathrm{~d})$ and $7.8(\mathrm{f})$ show the results of the simulation using Controller2. Figure $7.8(\mathrm{~b})$ shows that the distance errors decrease and stabilize to zero in much shorter time than using Controller-1. Comparing the charge histories in Figures 7.8(c) and $7.8(\mathrm{~d})$, it can be seen that there are more spikes when Controller- 2 is being used. Figure 7.8(b) also shows that the distance errors do not converge to zero. This is because the new switched control Controller-2 is stable, but not asymptotically stable.

It's not always the case that Controller-2 performs better than Controller-1. Under different initial conditions and different controller coefficients Controller-1 may perform better than Controller-2. There is one difference between the two simulation cases.


Figure 7.8: Small control effort simulations.

When using Controller-1 in the large control effort case in the illustrated simulation results, the distance errors settles down to a certain level and stack there. But in the small control effort case, the distance errors keep changing and won't stay at a certain level.

### 7.5 Conclusion

This chapter investigates the three-spacecraft Coulomb formation triangular shape control problem. Assuming continuous switching capability, a 2-side switched control strategy is developed to always control the worst two sides instead of controlling both of the three sides. Here an implementable control solution is always guaranteed. However, the discrete control time steps may cause temporary instability of the shape control. A stable switched control strategy is developed based on the multiple Lyapunov functions analysis. This new switch strategy ensures all of the error functions to be Lyapunov-like thus stability is guaranteed. Numerical simulations demonstrate the improvement of the stable switched control. The new switched control also induces spikes in the control charges because the new control changes the value of the distance rate feedback gain matrix to ensure stability. The method of employing Lyapunov-like control functions is a promising approach to investigate the relative control of charged spacecraft with more than three vehicles. The new switched control strategy is successful in stabilizing a non-equilibrium triangular shape.

## CHAPTER 8

## THREE-CRAFT EQUILIBRIUM FIXED SHAPE CONTROL

The previous chapter studies the control of a three-craft triangular shape configuration. For a general triangular shape, there does not exist an equilibrium charge solution, which makes the control problem very changeling. That is why the controller has to keep switching to control different two sides at every instance.

This chapter studies the special case of the three-craft Coulomb virtual structure control where the expected configuration is collinear. For a general collinear configuration, there always exists equilibrium charge solution. By this information, we know that the controller does not have to keep switching frequently to control only two sides of the system. In the neighborhood of the equilibrium, the controller should be capable to control both three sides simultaneously. From this perspective, the control charges will be smoother comparing to the stable switched control developed in the previous chapter. This is the motivation of studying the shape control specifically for the collinear configuration in this chapter.

### 8.1 Equations Of Motion

This chapter develops a control algorithm using only Coulomb forces to make a free-flying three-craft system stabilized to the desired collinear configuration. Figure 8.1 shows the setup of the spinning three-craft system. Figure 8.1(a) shows the basic geometry and notations. The parameters $m_{1}-m_{3}$ are the masses of the three spacecraft,


Figure 8.1: Charged three-Body system.
$\boldsymbol{r}_{1}-\boldsymbol{r}_{3}$ are the three inertial position vectors of the spacecraft, $\boldsymbol{r}_{12}-\boldsymbol{r}_{13}$ are the relative position vectors between the spacecraft, and $\boldsymbol{r}_{c m}$ is the inertial position vector of the center of mass (CM). Note that if we let the center of the inertial frame be the CM of the 3 -craft, then $\boldsymbol{r}_{c m}=\mathbf{0}$. The angle $\alpha_{i}$ is the angle between the two relative position vectors cornered at the $i^{\text {th }}$ spacecraft. Figure 8.1(b) shows the scenario of the equilibrium/expected state. The collinear configuration system is spinning about the CM. Note that depending on the CM location, the spinning direction of $m_{2}$ may be reversed from the illustration in Figure 8.1(b).

By the assumption that the three-craft system is flying in a free space, the inertial equations of motion (EOMs) of the three spacecraft are given by:

$$
\begin{align*}
& m_{1} \ddot{\boldsymbol{r}}_{1}=-k_{c} \frac{q_{1} q_{2}}{r_{12}^{2}} \hat{\boldsymbol{e}}_{12}-k_{c} \frac{q_{1} q_{3}}{r_{13}^{2}} \hat{\boldsymbol{e}}_{13}  \tag{8.1a}\\
& m_{2} \ddot{\boldsymbol{r}}_{2}=k_{c} \frac{q_{1} q_{2}}{r_{12}^{2}} \hat{\boldsymbol{e}}_{12}-k_{c} \frac{q_{2} q_{3}}{r_{23}^{2}} \hat{\boldsymbol{e}}_{23}  \tag{8.1b}\\
& m_{3} \ddot{\boldsymbol{r}}_{3}=k_{c} \frac{q_{1} q_{3}}{r_{13}^{2}} \hat{\boldsymbol{e}}_{13}+k_{c} \frac{q_{2} q_{3}}{r_{23}^{2}} \hat{\boldsymbol{e}}_{23} \tag{8.1c}
\end{align*}
$$

where $k_{c}=8.99 \times 10^{9} \mathrm{Nm}^{2} \mathrm{C}^{-2}$ is the Coulomb constant, $q_{i}$ is the charge of the $i^{\text {th }}$ spacecraft which can be actively controlled, $\hat{\boldsymbol{e}}_{i j}$ is the unit vector pointing from the $i^{\text {th }}$ to the $j^{\text {th }}$ spacecraft, and $r_{i j}$ is the separation distance between the $i^{\text {th }}$ and the $j^{\text {th }}$ spacecraft. These equations of motion assumes that the separation distance is small compared to the local plasma Debye length, and that the partial charge shielding from the plasma environment can be ignored. For the convenience of notation, a vector $\boldsymbol{\xi}$ is defined as a function of the charge products and the separation distances:

$$
\begin{equation*}
\boldsymbol{\xi}=\left(\xi_{1}, \xi_{2}, \xi_{3}\right)^{T}=\left(k_{c} \frac{q_{1} q_{2}}{r_{12}^{2}}, k_{c} \frac{q_{2} q_{3}}{r_{23}^{2}}, k_{c} \frac{q_{1} q_{3}}{r_{13}^{2}}\right)^{T} \tag{8.2}
\end{equation*}
$$

One way to specify the collinear configuration is through the relationship of the separation distances:

$$
\begin{equation*}
r_{13}=r_{12}+r_{23} \tag{8.3}
\end{equation*}
$$

Note that the separation distances can be directly controlled using only Coulomb forces, the separation distances' EOMs are expected for developing the control algorithm. Using the inertial EOMs in Eq. (8.1), the following relative positions' EOMs are achieved:

$$
\begin{align*}
\ddot{\boldsymbol{r}}_{12} & =\ddot{\boldsymbol{r}}_{2}-\ddot{\boldsymbol{r}}_{1}=\xi_{1}\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right) \hat{\boldsymbol{e}}_{12}-\xi_{2} \frac{1}{m_{2}} \hat{\boldsymbol{e}}_{23}+\xi_{3} \frac{1}{m_{1}} \hat{\boldsymbol{e}}_{13}  \tag{8.4a}\\
\ddot{\boldsymbol{r}}_{23} & =\ddot{\boldsymbol{r}}_{3}-\ddot{\boldsymbol{r}}_{2}=-\xi_{1} \frac{1}{m_{2}} \hat{\boldsymbol{e}}_{12}+\xi_{2}\left(\frac{1}{m_{2}}+\frac{1}{m_{3}}\right) \hat{\boldsymbol{e}}_{23}+\xi_{3} \frac{1}{m_{3}} \hat{\boldsymbol{e}}_{13}  \tag{8.4b}\\
\ddot{\boldsymbol{r}}_{13} & =\ddot{\boldsymbol{r}}_{3}-\ddot{\boldsymbol{r}}_{1}=\xi_{1} \frac{1}{m_{1}} \hat{\boldsymbol{e}}_{12}+\xi_{2} \frac{1}{m_{3}} \hat{\boldsymbol{e}}_{23}+\xi_{3}\left(\frac{1}{m_{1}}+\frac{1}{m_{3}}\right) \hat{\boldsymbol{e}}_{13} \tag{8.4c}
\end{align*}
$$

Applying the relationship between the separation distance and the relative position vector $\ddot{r}_{i j}=\ddot{\boldsymbol{r}}_{i j} \cdot \hat{\boldsymbol{e}}_{i j}+\frac{1}{r_{i j}}\left\|\dot{\boldsymbol{r}}_{i j}\right\|^{2}\left(1-\cos ^{2} \angle\left(\boldsymbol{r}_{i j}, \dot{\boldsymbol{r}}_{i j}\right)\right)$ to Eq. (8.4), yields the separation distances' EOMs:

$$
\begin{align*}
& \ddot{r}_{12}=\xi_{1}\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right)+\xi_{2} \frac{1}{m_{2}} \cos \alpha_{2}+\xi_{3} \frac{1}{m_{1}} \cos \alpha_{1}+g_{1}  \tag{8.5a}\\
& \ddot{r}_{23}=\xi_{1} \frac{1}{m_{2}} \cos \alpha_{2}+\xi_{2}\left(\frac{1}{m_{2}}+\frac{1}{m_{3}}\right)+\xi_{3} \frac{1}{m_{3}} \cos \alpha_{3}+g_{2}  \tag{8.5b}\\
& \ddot{r}_{13}=\xi_{1} \frac{1}{m_{1}} \cos \alpha_{1}+\xi_{2} \frac{1}{m_{3}} \cos \alpha_{3}+\xi_{3}\left(\frac{1}{m_{1}}+\frac{1}{m_{3}}\right)+g_{3} \tag{8.5c}
\end{align*}
$$

where $g_{i}$ is the shortcut for the highly nonlinear term:

$$
\begin{align*}
& g_{1}=\frac{1}{r_{12}}\left\|\dot{\boldsymbol{r}}_{12}\right\|^{2}\left(1-\cos ^{2} \angle\left(\boldsymbol{r}_{12}, \dot{\boldsymbol{r}}_{12}\right)\right)  \tag{8.6a}\\
& g_{2}=\frac{1}{r_{23}}\left\|\dot{\boldsymbol{r}}_{23}\right\|^{2}\left(1-\cos ^{2} \angle\left(\boldsymbol{r}_{23}, \dot{\boldsymbol{r}}_{23}\right)\right)  \tag{8.6b}\\
& g_{3}=\frac{1}{r_{13}}\left\|\dot{\boldsymbol{r}}_{13}\right\|^{2}\left(1-\cos ^{2} \angle\left(\boldsymbol{r}_{13}, \dot{\boldsymbol{r}}_{13}\right)\right) \tag{8.6c}
\end{align*}
$$

### 8.2 Equilibrium Charge Solution

The objective of the control development is to find an algorithm of the charges $\left[q_{1}, q_{2}, q_{3}\right]^{T}$ that makes the separation distances stabilized to the desired distances

$$
\left[r_{12}, r_{23}, r_{13}\right]^{T} \rightarrow\left[r_{12}^{*}, r_{23}^{*}, r_{13}^{*}\right]^{T}
$$

where the following requirement enforces the expected configuration to be collinear:

$$
\begin{equation*}
r_{13}^{*}=r_{12}^{*}+r_{23}^{*} \tag{8.7}
\end{equation*}
$$



Figure 8.2: Geometries of the equilibrium state.

This section investigates the equilibrium charge solutions under given expected separation distances. There are two questions need to be answered: does the solution exist; what is(are) the solution(s). For easier understanding of the physical meanings of $g_{i}$, let us rewrite these nonlinear terms in terms of the angular momentum:

$$
\boldsymbol{g}^{*}=\left[\begin{array}{c}
g_{1}^{*}  \tag{8.8}\\
g_{2}^{*} \\
g_{3}^{*}
\end{array}\right]=\left[\begin{array}{c}
r_{12}^{*} \frac{H^{2}}{\left(\Sigma m_{i} r_{i}^{2}\right)^{2}} \\
r_{23}^{*} \frac{H^{2}}{\left(\Sigma m_{i} r_{i}^{2}\right)^{2}} \\
r_{13}^{*} \frac{H^{2}}{\left(\Sigma m_{i} r_{i}^{2}\right)^{2}}
\end{array}\right]
$$

where $H$ is the magnitude of the angular momentum of the three-craft system $\boldsymbol{H}=$ $\Sigma \boldsymbol{r}_{i} \times\left(m_{i} \dot{\boldsymbol{r}}_{i}\right)$. Because there are no external forces acting on the system, the angular momentum is conserved, thus $H$ is determined by the initial conditions. The scalar parameter $r_{i}$ is the separation distance of the $i^{\text {th }}$ spacecraft measured from the CM as shown in Figure 8.2. The relationship between $r_{i}$ and $r_{i j}$ is:

$$
\left\{\begin{align*}
r_{1} & =\frac{m_{2}+m_{3}(1+a)}{m_{1}+m_{2}+m_{3}} r_{12}^{*}  \tag{8.9}\\
r_{2} & =r_{1}-r_{12}^{*} \\
r_{3} & =r_{1}-(1+a) r_{12}^{*}
\end{align*}\right.
$$

where $a$ is the ratio between the two separation distances $a=r_{23}^{*}: r_{12}^{*}$. Let us define other two ratio parameters as:

$$
\begin{equation*}
\beta=\frac{q_{2}^{*}}{q_{1}^{*}}, \quad \gamma=\frac{q_{3}^{*}}{q_{1}^{*}} \tag{8.10}
\end{equation*}
$$

Thus, instead of solving the individual charges $\left[q_{1}^{*}, q_{2}^{*}, q_{3}^{*}\right]^{T}$ directly, the following derivation solves the parameters $\left[q_{1}^{*}, \beta, \gamma\right]$ corresponding to the individual charges. The sepa-
ration distances' EOMs at the equilibrium state are derived from Eq. (8.5):

$$
\begin{align*}
& \ddot{r}_{12}^{*}=\frac{k_{c} q_{1}^{* 2}}{r_{12}^{* 2}}\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right) \beta-\frac{k_{c} q_{1}^{* 2}}{r_{12}^{* 2}} \frac{1}{m_{2}} \frac{\beta \gamma}{a^{2}}+\frac{k_{c} q_{1}^{* 2}}{r_{12}^{* 2}} \frac{1}{m_{1}} \frac{\gamma}{(1+a)^{2}}+g_{1}^{*}=0  \tag{8.11a}\\
& \ddot{r}_{23}^{*}=-\frac{k_{c} q_{1}^{* 2}}{r_{12}^{* 2}} \frac{1}{m_{2}} \beta+\frac{k_{c} q_{1}^{* 2}}{r_{12}^{* 2}}\left(\frac{1}{m_{2}}+\frac{1}{m_{3}}\right) \frac{\beta \gamma}{a^{2}}+\frac{k_{c} q_{1}^{* 2}}{r_{12}^{* 2}} \frac{1}{m_{3}} \frac{\gamma}{(1+a)^{2}}+a g_{1}^{*}=0  \tag{8.11b}\\
& \ddot{r}_{13}^{*}=\frac{k_{c} q_{1}^{* 2}}{r_{12}^{* 2}} \frac{1}{m_{1}} \beta+\frac{k_{c} q_{1}^{* 2}}{r_{12}^{* 2}} \frac{1}{m_{3}} \frac{\beta \gamma}{a^{2}}+\frac{k_{c} q_{1}^{* 2}}{r_{12}^{* 2}}\left(\frac{1}{m_{1}}+\frac{1}{m_{3}}\right) \frac{\gamma}{(1+a)^{2}}+(1+a) g_{1}^{*}=0 \tag{8.11c}
\end{align*}
$$

Next this chapter is going to find the solutions for $\left[q_{1}^{*}, \beta, \gamma\right]$ from Eq. (8.11). Note that the three equations in Eq. (8.11) are linearly coupled, specifically adding up Eq. (8.11a) and Eq. (8.11b) yields Eq. (8.11c). Thus there are actually two independent equations for solving three parameters. There exists one extra DOF in this problem. Supposing that a value $q_{1}^{*}$ is given, performing some algebras of Eq. (8.11), yields the following quadratic equation for $\gamma$ :

$$
\begin{equation*}
\gamma^{2}+\gamma\left[(1+a)^{2} \nu\left(m_{1} m_{2}+(1+a) m_{1} m_{3}\right)\right]-(1+a)^{2} a^{2} \nu\left((1+a) m_{1} m_{3}+a m_{2} m_{3}\right)=0 \tag{8.12}
\end{equation*}
$$

where $\nu$ is a function of $q_{1}^{*}$ :

$$
\begin{equation*}
\nu=\frac{r_{12}^{* 3} H^{2}}{k_{c} q_{1}^{* 2}\left(\Sigma m_{i} r_{i}^{2}\right)^{2} \Sigma m_{i}} \tag{8.13}
\end{equation*}
$$

The other parameter $\beta$ is given by

$$
\begin{equation*}
\beta=-\frac{\gamma}{(1+a)^{2}}-\nu\left(m_{1} m_{2}+(1+a) m_{1} m_{3}\right) \tag{8.14}
\end{equation*}
$$

Thus, after solving $\gamma$ from Eq. (8.12), the equilibrium charge solution is solved. Note that the real solution to a quadratic equation does not always exist. The existence of the real solutions to Eq. (8.12) requires the following inequality:

$$
\begin{align*}
& f(\nu)=(1+a)^{4} m_{1}^{2}\left(m_{2}+(1+a) m_{3}\right)^{2} \nu^{2} \\
& \quad+2 a^{2}(1+a)^{2}\left((1+a) m_{1} m_{3}+2 a m_{2} m_{3}-m_{1} m_{2}\right) \nu+a^{4} \geq 0 \tag{8.15}
\end{align*}
$$

where $f(\nu)$ is a quadratic function of $\nu$. Note that $\nu$ is a function of $q_{1}^{*}$ as shown in Eq. (8.13), proper value of $q_{1}^{*}$ might guarantee the inequality in Eq. (8.15) to be true. For the inequality in Eq. (8.15), there are two cases need to be discussed:
(1) There are no real solutions or there are two identical solutions to $f(\nu)=0$. In this case the inequality in Eq. (8.15) is always true. $m_{1} \geq a m_{2}$ ensures this case.
(2) There are two distinct solutions to $f(\nu)=0$, corresponding to the situation $m_{1}<a m_{2}$. In this case the requirement for $\nu$ is $\nu \geq \nu_{2}$ or $\nu \leq \nu_{1}$, where $\nu_{1,2}$ are the two real solutions to $f(\nu)=0$ and $\nu_{2}>\nu_{1}$.

If $\nu_{2} \leq 0$, any choice of $q_{1}^{*}$ will automatically satisfy $\nu \geq \nu_{2}$, because $\nu>0$ by definition. Otherwise $\nu_{2}>0$, the requirement for $q_{1}^{*}$ is

$$
\begin{align*}
q_{1}^{*} & \leq \sqrt{\frac{r^{* 3} H^{2}}{k_{c}\left(\Sigma m_{i} r_{i}^{2}\right)^{2} \Sigma m_{i} \nu_{2}}}  \tag{8.16}\\
\text { or } \quad q_{1}^{*} \geq \sqrt{\frac{r^{* 3} H^{2}}{k_{c}\left(\Sigma m_{i} r_{i}^{2}\right)^{2} \Sigma m_{i} \nu_{1}}} & \text { if } \nu_{1}>0 . \tag{8.17}
\end{align*}
$$

The requirement in Eq. (8.16) can always be satisfied regardless of the charge saturation issue.

Concluding the above analysis of the existence of the equilibrium solutions, if $m_{1}>a m_{2}$ or $\nu_{2} \leq 0$, given any value of $q_{1}^{*}$ there exists a pair of equilibrium solutions. Otherwise any value of $q_{1}^{*}$ that satisfies the inequality in Eq. (8.16) results in a pair of equilibrium solutions.

### 8.3 Lyapunov-Based Nonlinear Control Algorithm

The objective of the control is to make the three separation distances stabilized to the desired separation distances corresponding to the desired collinear configuration. Define the state vector as:

$$
\begin{equation*}
\boldsymbol{X}=\left(r_{12}, r_{23}, r_{13}\right)^{T} \tag{8.18}
\end{equation*}
$$

The objective is rephrased as $\boldsymbol{X} \rightarrow \boldsymbol{X}^{*}$ where $\boldsymbol{X}^{*}$ represents the desired separation distances. The separation distances' EOMs is rewritten as:

$$
\begin{equation*}
\ddot{\boldsymbol{X}}=[B] \boldsymbol{\xi}+\boldsymbol{g} \tag{8.19}
\end{equation*}
$$

where $[B]$ is the $3 \times 3$ matrix:

$$
[B]=\left[\begin{array}{ccc}
\frac{1}{m_{1}}+\frac{1}{m_{2}} & \frac{\cos \alpha_{2}}{m_{2}} & \frac{\cos \alpha_{1}}{m_{1}}  \tag{8.20}\\
\frac{\cos \alpha_{2}}{m_{2}} & \frac{1}{m_{2}}+\frac{1}{m_{3}} & \frac{\cos \alpha_{3}}{m_{3}} \\
\frac{\cos \alpha_{1}}{m_{1}} & \frac{\cos \alpha_{3}}{m_{3}} & \frac{1}{m_{1}}+\frac{1}{m_{3}}
\end{array}\right]
$$

Note that $\boldsymbol{\xi}$ is a function of the individual charges. The following development finds an implementable algorithm of $\boldsymbol{\xi}$ to stabilize the system.

Define the Lyapunov function to be:

$$
\begin{equation*}
V=\frac{1}{2} \Delta \boldsymbol{X}^{T}[K] \Delta \boldsymbol{X}+\frac{1}{2} \Delta \dot{\boldsymbol{X}}^{T} \Delta \dot{\boldsymbol{X}} \tag{8.21}
\end{equation*}
$$

where $[K]$ is a $3 \times 3$ positive definite function. Taking a time derivative of $V$ and utilizing the EOMs in Eq. (8.19), yields:

$$
\begin{align*}
\dot{V} & =\Delta \dot{\boldsymbol{X}}([K] \Delta \boldsymbol{X}+\Delta \ddot{\boldsymbol{X}}) \\
& =\Delta \dot{\boldsymbol{X}}([K] \Delta \boldsymbol{X}+[B] \boldsymbol{\xi}+\boldsymbol{g}) \tag{8.22}
\end{align*}
$$

In order to utilize the equilibrium charge solution developed in the last section, the control vector $\boldsymbol{\xi}$ is rewritten as:

$$
\begin{equation*}
\boldsymbol{\xi}=\boldsymbol{\xi}^{*}+\delta \boldsymbol{\xi} \tag{8.23}
\end{equation*}
$$

where $\boldsymbol{\xi}^{*}$ corresponds to the equilibrium charge solution. Forcing $\dot{V}$ to be the negative semi-definite form:

$$
\begin{equation*}
\dot{V}=-\Delta \dot{\boldsymbol{X}}^{T}[P] \Delta \dot{\boldsymbol{X}} \tag{8.24}
\end{equation*}
$$

yields the following equation:

$$
\begin{equation*}
[B] \delta \boldsymbol{\xi}=-[P] \Delta \dot{\boldsymbol{X}}-[K] \Delta \boldsymbol{X}-[B] \boldsymbol{\xi}^{*}-\boldsymbol{g} \tag{8.25}
\end{equation*}
$$

Solving $\delta \boldsymbol{\xi}$ from Eq. (8.25) yields a solution of $\boldsymbol{\xi}$ that would stabilize the system. One issue is that by the definition in Eq. (8.20) the matrix $[B]$ is not always invertible. The following theorem proves that the matrix $[B]$ is singular only at the collinear configurations.

Theorem 10 Given a $3 \times 3$ matrix [ $B$ ] defined by Eq. (8.20), with ( $\alpha_{1}, \alpha_{2}, \alpha_{3}$ ) being the inner angles of a triangle as shown in Figure 8.1(a), the matrix $[B]$ is singular if and only if $\cos \alpha_{1} \cos \alpha_{2} \cos \alpha_{3}=-1$.

Proof One necessary and sufficient condition for a square matrix to be singular is that its determinant is zero. To prove the above theorem, we just need to prove that $|[B]|=0$ if and only if (iff.) $\cos \alpha_{1} \cos \alpha_{2} \cos \alpha_{3}=-1$. From the definition in Eq. (8.20), the determinant of $[B]$ is

$$
\begin{align*}
|[B]|=( & \left.\frac{1}{m_{1}}+\frac{1}{m_{2}}\right)\left(\frac{1}{m_{2}}+\frac{1}{m_{3}}\right)\left(\frac{1}{m_{1}}+\frac{1}{m_{3}}\right)+\frac{2 \cos \alpha_{1} \cos \alpha_{2} \cos \alpha_{3}}{m_{1} m_{2} m_{3}} \\
& -\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right) \frac{\cos ^{2} \alpha_{3}}{m_{3}^{2}}-\left(\frac{1}{m_{2}}+\frac{1}{m_{3}}\right) \frac{\cos ^{2} \alpha_{1}}{m_{1}^{2}}-\left(\frac{1}{m_{1}}+\frac{1}{m_{3}}\right) \frac{\cos ^{2} \alpha_{2}}{m_{2}^{2}} \tag{8.26}
\end{align*}
$$

Because $\cos ^{2} \alpha_{i} \leq 1$, the following inequality is true:

$$
\begin{align*}
&|[B]| \geq\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right)\left(\frac{1}{m_{2}}+\frac{1}{m_{3}}\right)\left(\frac{1}{m_{1}}+\frac{1}{m_{3}}\right)+\frac{2 \cos \alpha_{1} \cos \alpha_{2} \cos \alpha_{3}}{m_{1} m_{2} m_{3}} \\
&-\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right) \frac{1}{m_{3}^{2}}-\left(\frac{1}{m_{2}}+\frac{1}{m_{3}}\right) \frac{1}{m_{1}^{2}}-\left(\frac{1}{m_{1}}+\frac{1}{m_{3}}\right) \frac{1}{m_{2}^{2}} \tag{8.27}
\end{align*}
$$

Note that the equal relationship in Eq. (8.27) is true iff. $\cos ^{2} \alpha_{i}=1, i=1,2,3$. The RHS. of the inequality can be simplified:

$$
\begin{equation*}
|[B]| \geq \frac{2+2 \cos \alpha_{1} \cos \alpha_{2} \cos \alpha_{3}}{m_{1} m_{2} m_{3}} \tag{8.28}
\end{equation*}
$$

From the inequality in Eq. (8.27), the determinant $|[B]| \geq 0,|[B]|=0$ iff. $\cos \alpha_{1} \cos \alpha_{2} \cos \alpha_{3}=$ -1 .

Note that the necessary and sufficient condition for $|[B]|=0, \cos \alpha_{1} \cos \alpha_{2} \cos \alpha_{3}=$ -1 , represents all the collinear configurations, including the desired collinear configuration specified by $\boldsymbol{X}^{*}$.

Let us investigate those situations where $|[B]|=0$. At the equilibrium state where $\boldsymbol{X}=\boldsymbol{X}^{*}$, the state errors $\Delta \boldsymbol{X}=0$ and $\Delta \dot{\boldsymbol{X}}=0$. From the procedure of finding
the equilibrium charge solution, at the equilibrium state $[B] \boldsymbol{\xi}^{*}+\boldsymbol{g}=0$. Thus at the equilibrium state, the RHS of Eq. (8.25) is zero. In this situation, any element in the null space of the singular matrix $[B]$ is the solution to Eq. (8.25), including the zero vector.

Another special case is the one-dimensional constraint motion. This situation has been discussed in Chapter 6. In this case there are only two DOFs in the system, we need to control only two sides. Thus we use only two equations from Eq. (8.25), which contains three equations.

Other than the above two cases, we can not control all the three sides simultaneously when $[B]$ is singular. In this situation, the stable switched control strategy developed in Chapter 7 is engaged to control only the two sides of the system at each intance while the stability is still guaranteed.

### 8.4 Numerical Simulations

### 8.4.1 Control Results With The Exact Feedforward Part

Figure 8.3 shows the simulation results in the case that the feedforward charge components are calculated using the correct value of the angular momentum of the 3 -craft system. The initial conditions are as following:

$$
\begin{align*}
\boldsymbol{R}_{1} & =[-4,1,0]^{T} \mathrm{~m}  \tag{8.29}\\
\boldsymbol{R}_{2} & =[25,0,0]^{T} \mathrm{~m}  \tag{8.30}\\
\boldsymbol{R}_{3} & =[40,0,0]^{T} \mathrm{~m}  \tag{8.31}\\
\dot{\boldsymbol{R}}_{1} & =[0,0.001,0.0001]^{T} \mathrm{~m} / \mathrm{s}  \tag{8.32}\\
\dot{\boldsymbol{R}}_{1} & =[0,0,0]^{T} \mathrm{~m} / \mathrm{s}  \tag{8.33}\\
\dot{\boldsymbol{R}}_{1} & =[0,0.001,0]^{T} \mathrm{~m} / \mathrm{s} \tag{8.34}
\end{align*}
$$

The masses of the spacecraft are $m_{1}=m_{2}=m_{3}=50 \mathrm{~kg}$. The controller coeffi-


Figure 8.3: Simulation results with the exact knowledge of the angular momentum.
cients are $[P]=0.00015 I_{3 \times 3} \mathrm{~s}^{-1},[K]=10^{-8} I_{3 \times 3} \mathrm{~s}^{-2}$, where $I_{3 \times 3}$ is the $3 \times 3$ identity matrix. The expected separation distances are set to be $\boldsymbol{X}^{*}=[20,20,40]^{T} \mathrm{~m}$.

Figure 8.3(a) shows the trajectories as seen from the inertial frame, the boxes represent the final locations of the three spacecraft. Figure 8.3(b) shows the histories of the separation distances. Figure 8.3(c) shows the separation distance errors in log scale. The threshold for turning off the feedback part is $V=10^{-11} \mathrm{~m}^{2} / \mathrm{s}^{2}$. In this case the distance error levels settle down to centimeter level. Figure 8.3(d) shows the charge histories. There are three situations of the control: 3 -side control when $[B]$ is invertible; 2-side switched control when $[B]$ is not invertible; only feedforward part when $V<10^{-11} \mathrm{~m}^{2} / \mathrm{s}^{2}$.

### 8.4.2 Simulation Results With Incorrect Feedforward Part

This simulation example tests the case that the angular momentum information is incorrect thus results in an incorrect set of the equilibrium charges. The initial conditions are exactly the same as the previous simulation. In calculating the equilibrium charges, the angular momentum is reduced to $20 \%$ of the true value $H_{e}=0.2 H$. The resulting feedforward charges are $\boldsymbol{q}_{f f}=[1,-0.255,1] \mu \mathrm{C}$. But corresponding to the true value of the angular momentum and $q_{e 1}=1 \mu \mathrm{C}$, the equilibrium charges should be $\boldsymbol{q}_{e}=[1,-0.385,1] \mu \mathrm{C}$. This indicates that the feedforward charges are not the equilibrium charges, without the feedback part the system will not stay at the equilibrium configuration.

The simulation results are shown in Figure 8.4. Comparing to Figure 8.3, one can find that the response of the system is similar, except after around 28 hours of the simulation time. That is because at that time the feedback part of the control is turned off because $V<10^{-11} \mathrm{~m}^{2} / \mathrm{s}^{2}$. But since the feedforward charges are not the equilibrium charges, the control charges drive the system worse much faster than the previous simulation where the exact feedforward charges are achieved. From the charge histories in Figure 8.4(d), it can be seen that the time span when the feedback charges are turned off is much shorter than that in the previous simulation. This explains why the response of the system is different from the previous simulation after around 28 hours of the simulation time.

### 8.4.3 Extreme Simulation Example

This simulation case is to use an extreme example to illustrate that the control works globally. The "extreme" example is chosen to be that the controller needs to "flip-over" the locations of two spacecraft. All the simulation conditions are the same as the previous simulations except that the locations of the three spacecraft are set to


Figure 8.4: Simulation results without the exact knowledge of the angular momentum.
be:

$$
\begin{align*}
& \boldsymbol{R}_{1}=[-4,1,0]^{T} \mathrm{~m}  \tag{8.35}\\
& \boldsymbol{R}_{2}=[40,0,0]^{T} \mathrm{~m}  \tag{8.36}\\
& \boldsymbol{R}_{3}=[25,0,0]^{T} \mathrm{~m} \tag{8.37}
\end{align*}
$$

In this set of initial conditions, SC-3 is roughly allocated in the center of the nearly collinear configuration. While the expected collinear configuration requires SC-2 be the center of the configuration, the controller needs to flip-over the relative positions of SC-2 and SC-3.

The simulation results are shown in Figure 8.5. The trajectories in Figure 8.5(a) seems a bit nasty, but the final positions marked by the three boxes shows the three


Figure 8.5: Simulation results without the exact knowledge of the angular momentum.
spacecraft are alined. Figure 8.5(b) shows that $L_{23}$ and $L_{13}$ stabilize to their expected values after around 10 hours, $L_{12}$ grows up to 60 m , then stabilizes after around 40 hours. The distance errors in Figure 8.5(c) show that the error of $L_{12}$ stabilizes to centimeter level, while the errors in $L_{23}$ and $L_{13}$ are at much lower level. Figure 8.5(d) shows that there is a spike of the control charges goes up to $80 \mu$ C. Practically this charge level is very difficult to achieve using current technology. This simulation example is just used to illustrate the theoretical global-ness of the control algorithm.

### 8.5 Conclusion

This section develops a Lyapunov-based globally stable control algorithm to make a three-craft Coulomb formation stabilized to an expected collinear configuration. The
equilibrium charge solution is utilized as the feedforward part of the control. The stable switched control strategy is engaged when it is not implementable to control both three sides simultaneously. Numerical simulations show that even when the feedforward charges are incorrect the control still stabilizes the system to the desired configuration. An extreme simulation example, which requires to flip-over the relative positions of two spacecraft, illustrate that the control is globally stable.

A comparison to the charge modulation approach is an interesting future work of the three-craft Coulomb virtual structure control. Charge modulation approach is another way to solve the implementation issue. Instead of controlling both sides during one control cycle, the charge modulation method continuously divides the cycle into several sub-cycles. During each sub-cycle, this approach generates charges to control only a certain subset of all of the three sides, and the charge levels are several times greater than that needed for one control cycle. Because the charge modulation approach continuously switches between the sub- sets at a constant frequency, the power consumption is much greater. Another obvious drawback of charge modulation method is that, as the number of spacecraft in the cluster increases, the sub-cycle period becomes very small. This means the charge modulation must work as a very high frequency if the Coulomb structure is composed of many spacecraft. This high frequency not only increases power consumption, it might not be able to achieved at all.

## CHAPTER 9

## CONCLUSIONS

Based on the point-mass, point-charge model, this dissertation studies the application of Coulomb thrusting in the two-craft collision avoidance problem and the Coulomb virtual structure control problem. In both of these studies, the dissertation uses only Coulomb forces as the control input.

For the spacecraft collision avoidance problem, the Lyapunov-based feedback control achieves collision avoidance and retains the relative kinetic energy level. The openloop trajectory programming algorithm achieves collision avoidance and retains both the direction and the magnitude of the relative velocity. The trajectory programming algorithm utilizes the extra degree of freedom to find an optimal trajectory according to a certain cost function.

Regarding the Coulomb virtual structure control problem, the study of the spinning two-craft formation control in GEO orbit reveals that, the spinning of the two-craft system dominates the gravitational part in the closed-loop dynamics assuming a fast spin rate, and the influence of the spinning is stable. For a three-craft Coulomb virtual structure, there do not exist equilibrium charge solutions except for the collinear configuration. For the triangular configuration control problem, always controlling both three sides is not possible. The dissertation presents a stable switched two-side control strategy to ensure the existence of the null space of the control input matrix, which can be utilized to find implementable solutions. With the presence of the control input dis-
cretization, the stability of the switched control is ensured using the multiple Lyapunov function analysis tool.

Concluding the contributions, this dissertation broadens the application of Coulomb thrusting. It investigates the application of Coulomb thrusting in spacecraft collision avoidance problem and nonlinear control of Coulomb virtual structure. Technical wise, the following results/ideas are valuable for future research in Coulomb formation flying area:
(1) The patched conic-section programming algorithm can be generalized to the flyby maneuver or gravity assist concept. The symmetric trajectory programming algorithm achieves a trajectory that retains the magnitude and direction of the relative velocity. By altering the constraints of the problem, the algorithm might be able to achieve an arbitrary relative velocity that corresponds to the flyby maneuver objective.
(2) The dissertation presents two approaches to find implementable charge solutions. The first one is to utilize the null space of the control input matrix. By utilizing the null space, the control solution can be modified to be implementable without changing the dynamics of the system. The second one is to vary the Lyapunov function rate. By varying the value of the Lyapunov function rate, the control solution is modified. Certain constraint must be satisfied to ensure control implementability and stability of the system. So this approach is more restrictive to apply than the first approach.
(3) In the three-craft triangular shape control problem, the strategy of controlling the "worst" two sides ensures the existence of the null space of the control input matrix. This idea can be generalized to be controlling a subset of the system, which can be utilized in multiple-craft shape control problem.

The first item in the above listed contributions provides a direction of future re-
search after this dissertation, Coulomb flyby maneuver. The second and third items provide important tools for general multiple-craft Coulomb virtual structure control problem. Besides these two directions of future research, note that the dissertation makes several simplifications in setting up the Coulomb virtual structure control problem. For example the dissertation does not take gravitational forces, Debye shielding effect and other disturbances into account in three-craft Coulomb virtual structure control part. Future research need to investigate the effects of these forces/disturbances to the stability of the system.

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[^0]:    ${ }^{1}$ http://www.psatellite.com/research/formationflying.php

[^1]:    ${ }^{1}\left\{\hat{\boldsymbol{\imath}}_{v}, \hat{\boldsymbol{\imath}}_{h}, \hat{\boldsymbol{\imath}}_{D}\right\}$ centers at SC 1 , with $\hat{\boldsymbol{\imath}}_{v}$ pointing to the SC2's relative velocity direction, $\hat{\boldsymbol{\imath}}_{h}$ is the unit vector of the relative angular momentum, $\hat{\boldsymbol{\imath}}_{D}$ closes the right hand coordinate.

